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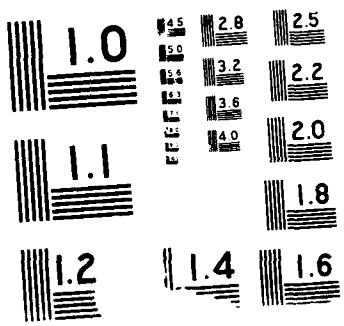
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CMS Technical Summary Report #88-23

PERIODIC SOLUTIONS OF PERTURBED SUPERQUADRATIC HAMILTONIAN SYSTEMS

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ABSTRACT

In this paper, we prove the existence of infinitely many distinct T-periodic solutions of the perturbed Hamiltonian system

$$\dot{z} = J(H'(z) + f(t))$$

under the conditions that H is of C^1, superquadratic, and possesses exponential or polynomial growth at infinity and that f is of $W^{1,2}$ and T-periodic, via minimax methods.

AMS (MOS) Subject Classifications: 34C25, 58E05, 58F05

Key Words: Hamiltonian systems, superquadratic, perturbation, monotone truncations, a priori estimates, minimax methods, multiple periodic solutions

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PERIODIC SOLUTIONS OF PERTURBED SUPERQUADRATIC

HAMILTONIAN SYSTEMS

Yiming Long

§1. Introduction and main results

We consider the existence of periodic solutions of a perturbed Hamiltonian system

(1.1)
$$\dot{z} = J(H'(z) + f(t))$$

where $z, f : \mathbf{R} \to \mathbf{R}^{2N}$, $\dot{z} = \frac{dz}{dt}$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I is the identity matrix on $\mathbf{R}^N$, $H : \mathbf{R}^{2N} \to \mathbf{R}$, H' is its gradient. Let $a \cdot b$ and $|\cdot|$ denote the usual inner product and norm on $\mathbf{R}^{2N}$. H will be required to satisfy the following conditions,

(H1) $H \in C^1(\mathbf{R}^{2N}, \mathbf{R})$.

(H2) There exist $\mu > 2$, $r_0 > 0$ such that $0 < \mu H(z) \le H'(z) \cdot z$, $\forall |z| \ge r_0$.

(H3) There exist $0 < \frac{q_2}{2} < q_1 \le q_2 < 2$ and $\alpha_i, \tau_i > 0$, $\beta_i \ge 0$ for $i = 1,2$ such that

$$\alpha_1 e^{\tau_1 |z|^{q_1}} - \beta_1 \le H(z) \le \alpha_2 e^{\tau_2 |z|^{q_2}} + \beta_2 \qquad \forall z \in \mathbf{R}^{2N}$$

or

(H4) There exist $1 < p_1 \le p_2 < 2p_1 + 1$, $\alpha_i > 0$, $\beta_2 \ge 0$ for $i = 1,2$ such that

$$\alpha_1 |z|^{p_1+1} - \beta_1 \le H(z) \le \alpha_2 |z|^{p_2+1} + \beta_2 \qquad \forall z \in \mathbf{R}^{2N} .$$

Our main results are

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Theorem 1.2. Let H satisfy conditions (H1)-(H3). Then for any given $T > 0$ and T-periodic function $f \in W^{1,2}_{loc}(R, R^{2N})$, (1.1) possesses an unbounded sequence of T-periodic solutions.

Theorem 1.3. The conclusion of Theorem 1.2 holds under conditions (H1), (H2) and (H4).

Such global existence problems have been studied extensively in recent years. For the autonomous case of (1.1) (i.e. $f \equiv 0$), the result was proved by Rabinowitz under the conditions (H1) and (H2) only ([17, 19]). His proof is based on a group symmetry possessed by the corresponding variational formulation. When one considers the forced vibration problem (1.1) (f does depend on t), such a symmetry breaks down. Bahri and Berestycki ([2]) studied the perturbed problem (1.1) and proved the conclusion of Theorem 1.3 by assuming (H2), (H4) and (H1') : $H \in C^2(R^{2N}, R)$. Our Theorem 1.3 weakens their condition on the smoothness of H and Theorem 1.2 allows H to increase faster at infinity.

Our proof extends Rabinowitz' basic ideas used in [18,19] and ideas used in [12,13]. In order to get the smoothness and compactness of corresponding functionals, we introduce a sequence of truncation functions $\{H_n\}$ of H in C^1 and corresponding modified functionals $\{J_n\}$ and J of I, where

(1.4) $$I(z) = \int_0^{2\pi} \left(\frac{1}{2}\dot{z}\cdot Jz - H(z) - f\cdot z\right)dt \quad .$$

We make $\{H_n\}$ be monotone increasing to H and $\{J_n\}$ be monotone decreasing to J as n increases. These monotonicities also allow us to get L^∞-estimates for the critical points of J_n we found. We modify the treatment of the S^1-action on $W^{1/2,2}(S^1, R^{2N})$ by introducing a simpler S^1-action on it to get upper estimates for certain minimax values. Combining with applications of Fadell-Rabinowitz cohomological index we get the multiple existence of periodic solutions of (1.1).

In §2 we define $\{H_n\}$, $\{J_n\}$ and J. With the aid of an auxiliary space X we define sequences of minimax values $\{a_k(n)\}$, $\{b_k(n)\}$ of J_n, $\{a_k\}$, $\{b_k\}$ of J in §3, and discuss their properties in §4. §5 and §6 contain estimates from above and below for $\{a_k\}$. We prove the existence of critical values of J_n in §7. Then in §8 by showing that the critical points of J_n for large n yields solutions of (1.1), we complete the proofs of our main theorems. Finally in §9 we discuss more general forced Hamiltonian systems.

Since the proofs of Theorems 1.2 and 1.3 are similar, we shall carry out the details for the first one only and make some comments on the second in §8. For the details of the proof of Theorem 1.3, we refer to [13].

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§2. Modified functionals

By rescaling time if necessary we can assume $T = 2\pi$. Let $E = W^{1/2,2}(S^1,\mathbb{R}^{2N})$. The scalar product in L^2 naturally extends as the duality pairing between E and $E' = W^{-1/2,2}(S^1,\mathbb{R}^{2N})$. Thus for $z \in E$, the actional integral $\frac{1}{2} A(z)$ is well defined, where

$$A(z) = \int_0^{2\pi} \dot{z}\cdot Jz\, dt \quad . \tag{2.1}$$

For $k \in \mathbb{N}$, write $\bar{k} = [\frac{k}{N}]$, $\tilde{k} = k - [\frac{k}{N}]$, where $[\frac{k}{N}]$ is the integer part of $\frac{k}{N}$. Let $e_1,\dots,e_{2N}$ denote the usual orthonormal basis in $\mathbb{R}^{2N}$. We write $i = \sqrt{-1}$, and denote for $k \in \mathbb{N}$

$$\begin{cases} \hat{\varphi}_k = (\sin \bar{k}t)e_{\tilde{k}} - (\cos \bar{k}t)e_{\tilde{k}+N} \quad , & i\hat{\varphi}_k = (\cos \bar{k}t)e_{\tilde{k}} + (\sin \bar{k}t)e_{\tilde{k}+N} \quad , \\ \hat{\psi}_k = (\sin \bar{k}t)e_{\tilde{k}} + (\cos \bar{k}t)e_{\tilde{k}+N} \quad , & i\hat{\psi}_k = (\cos \bar{k}t)e_{\tilde{k}} - (\sin \bar{k}t)e_{\tilde{k}+N} \quad . \end{cases} \tag{2.2}$$

Let $E^+_{m,n} = \mathrm{span}\{\hat{\varphi}_k, i\hat{\varphi}_k \mid m \leq k \leq n\}$, $E^-_{m,n} = \mathrm{span}\{\hat{\psi}_k, i\hat{\psi}_k \mid m \leq k \leq n\}$ for $m, n \in \mathbb{N}$, $N \leq m \leq n$. $E^+ = E^+_{N+1,+\infty}$, $E^- = E^-_{N+1,+\infty}$ and $E^0 = \mathrm{span}\{\hat{\varphi}_k, i\hat{\varphi}_k \mid 1 \leq k \leq N\}$. Then $E = E^+ \oplus E^- \oplus E^0$ and $A(z)$ is positive definite, negative definite and null on E^+, E^- and E^0 respectively. For $z = z^+ + z^- + z^0 \in E^+ \oplus E^- \oplus E^0 = E$, we take as a norm for E

$$\|z\|_E^2 \equiv A(z^+) - A(z^-) + |z^0|^2 \quad .$$

Under this norm, E becomes a Hilbert space and E^+, E^-, E^0 are orthogonal subspaces of E with respect to the inner product associated with this norm, as well as with the L^2 inner product. (2.2) gives a basis for E.

A result of Brezis and Wainger [6] implies that E is compactly embedded into $L^p(S^1,\mathbb{R}^{2N})$, $\forall\, 1 \leq p < +\infty$ and the Orlicz space L^*_M with $M(z) = e^{\tau|z|^q} - \sum_{k=0}^{n} \frac{\tau^k}{k!}|z|^{kq}$, $\forall\, \tau > 0$, $0 < q < 2$ and $n \in \mathbb{N}$, $nq > 1$, and $E \subset L_M$. Note that $q = 2$ is the critical imbedding value (cf. [6,10]). We shall use the following version of their imbedding theorem,

Lemma 2.3. For $\tau > 0$, $0 < q < 2$, there exist constants $C_1, C_2 > 0$ depending only on τ and q such that

$$\int_0^{2\pi} \exp(\sigma\tau|z|^q)dt \le C_1 \exp\left(C_2(\sigma\|z\|_E)^{\frac{q}{2-q}}\right) \qquad \forall\ \sigma > 0,\ z \in E\ .$$

Proof. We use the notations in [6] and only prove the Lemma for $E = W^{1/2,2}(S^1,\mathbb{C})$.

For $z \in E$, write $z(t) = C_0 + \sum_{n\neq 0} \frac{C_n}{\sqrt{|n|}} e^{int}$ where $C_n \in \mathbb{C}$. Then $z = k*g$, where $k(t) = \sum_{n\neq 0} \frac{1}{\sqrt{|n|}} e^{int} \in L(2,\infty)$, $g(t) = \sum_{n\in\mathbb{Z}} C_n e^{int} \in L(2,2)$. By (8) of [6],

$$\sigma\tau|z^*|^q \le (\varepsilon\sigma\tau)^{\frac{2}{q}}|z^*|^2 + \varepsilon^{-\frac{2}{2-q}}$$

$$\le (\varepsilon\sigma\tau)^{\frac{2}{q}}C(1 + |\log t|)\ \|k\|^2_{L(2,\infty)}\ \|g\|^2_{L(2,2)} + \varepsilon^{-\frac{2}{2-q}} \quad \text{for } 0 < t < 1\ .$$

Choose $\varepsilon = \frac{1}{\sigma\tau}\left(qC\|k\|^2_{L(2,\infty)}\|g\|^2_{L(2,2)}\right)^{-\frac{q}{2}}$, then from $\|g\|_{L(2,2)} \le C\|z\|_E$ and $\int_0^{2\pi} e^{\sigma\tau|z|^q}dt = \int_0^{2\pi} e^{\sigma\tau|z^*|^q}dt$, we get the lemma.

$\square$

For $z \in E$ and $\theta \in [0,2\pi] \simeq S^1$, we define an S^1-action on E by

$$(T_\theta z)(t) = z(t+\theta) \qquad \forall\ t \in [0,2\pi]\ .$$

We say a subset B of E is $S^1(E)$-invariant if $T_\theta z \in B$, $\forall\ z \in B$, $\theta \in [0,2\pi]$. Note that $\mathrm{Fix}\{T_\theta\} \equiv \{z \in E \mid T_\theta z = z,\ \forall\ \theta \in [0,2\pi]\} = E^0$.

By Lemma 2.3 and (H3), $I(z)$ defined in (1.4) is a continuous functional on E and formally the critical points of I correspond to the solutions of (1.1).

Without loss of generality we assume $r_0 > 1$. Set $\alpha_0 = \min_{|z|=r_0} H(z)$,

$\beta_0 = \beta_1 + \max_{|z|\le r_0} |H(z)|$, where β_1 is given by (H3). Conditions (H1) and (H2) imply that for some $\beta_3 > 0$,

$$(2.4)\quad \begin{cases} \alpha_0|z|^\mu \le H(z) \quad , \quad \forall\ |z| \ge r_0 \\ \alpha_0|z|^\mu \le H(z) + \beta_0 \le \dfrac{1}{\mu}\,(H'(z)\cdot z + \beta_3) \quad , \quad \forall\ z \in \mathbf{R}^{2N} . \end{cases}$$

Choose $\sigma \in (0,1)$ such that $\mu\sigma > 2$. We have

Proposition 2.5. Assume conditions (H1) and (H2). Then there exists a sequence $\{K_n\} \subset \mathbf{R}$ and a sequence of functions $\{H_n\}$ such that

1°. $0 < K_0 < K_n < K_{n+1}$, $\forall\ n \in \mathbf{N}$ and $K_n \to +\infty$ as $n \to +\infty$ where $K_0 = \max\left\{1, r_0, \dfrac{\beta_0}{\alpha_0(1-\sigma)}\right\}$.

2°. $H_n \in C^1(\mathbf{R}^{2N},\mathbf{R})$ $\quad \forall\ n \in \mathbf{N}$.

3°. $H_n(z) = H(z)$ $\quad \forall\ n \in \mathbf{N}$ and $|z| \le K_n$.

4°. $H_n(z) \le H_{n+1}(z) \le H(z)$ $\quad \forall\ n \in \mathbf{N}$ and $z \in \mathbf{R}^{2N}$.

5°. $0 < \mu\sigma H_n(z) \le H_n'(z)\cdot z$, $\forall\ n \in \mathbf{N}$ and $|z| \ge r_0$.

6°. For $n \in \mathbf{N}$, there exists a constant $\lambda_0 > 1$ independent of n and $C(n) > 0$ such that

$$|H_n'(z)|^{\lambda_0} \le C(n)(H_n'(z)\cdot z + 1) \qquad \forall\ z \in \mathbf{R}^{2N} .$$

Since the proof of this proposition is rather technical and lengthy, we put it in the Appendix.

Similar to (2.4), there is a constant $\beta_4 > 0$ independent of n such that

$$(2.6)\quad \begin{cases} \alpha_0|z|^{\mu\sigma} \le H_n(z) \qquad \forall\ n \in \mathbf{N} \text{ and } |z| \ge r_0 \\ \alpha_0|z|^{\mu\sigma} \le H_n(z) + \beta_0 \le \dfrac{1}{\mu\sigma}\,(H_n'(z)\cdot z + \beta_4) \qquad \forall\ n \in \mathbf{N} \text{ and } z \in \mathbf{R}^{2N} . \end{cases}$$

For $n \in \mathbf{N}$, we define

$$I_n(z) = \frac{1}{2}\,A(z) - \int_0^{2\pi} H_n(z)dt - \int_0^{2\pi} f\cdot z dt \qquad \forall\ z \in E .$$

Lemma 2.7.

1°. $I_n \in C^1(E,\mathbf{R})$, $\forall\, n \in \mathbf{N}$.

2°. $I(z) \leq I_{n+1}(z) \leq I_n(z)$, $\forall\, n \in \mathbf{N}$ and $z \in E$.

Proof: For 1° we refer to [5,17]. 2° follows from 4° of Proposition 2.5.

□

From now on in this section, we define $\tau = \frac{1}{4}\sqrt{3\mu\sigma+10}$.

Lemma 2.8. There is $\beta_5 > 0$ independent of n such that

$$(2.9)\qquad \frac{\tau(3\mu\sigma+2)}{2(\tau-1)}\|f\|_{L^2}\|z\|_{L^2} \leq \frac{3\mu\sigma+2}{8}\int_0^{2\pi}(H_n(z)+\beta_0)dt + \beta_5 \quad \forall\, n \in \mathbf{N} \text{ and } z \in E.$$

Proof. By (2.6)

$$\frac{3\mu\sigma+2}{8}\int_0^{2\pi}(H_n(z)+\beta_0)dt - \frac{\tau(3\mu\sigma+2)}{2(\tau-1)}\|f\|_{L^2}\|z\|_{L^2}$$

$$\geq \frac{3\mu\sigma+2}{8}\alpha_0\|z\|^{\mu\sigma}_{L^{\mu\sigma}} - \frac{\tau(3\mu\sigma+2)}{2(\tau-1)}\|f\|_{L^2}(2\pi)^{\frac{\mu\sigma-2}{2\mu\sigma}}\|z\|_{L^{\mu\sigma}}.$$

Since $\mu\sigma > 2$, (2.9) holds.

□

Lemma 2.10. There is $\beta_6 > 0$ independent of n such that for any $n \in \mathbf{N}$, $z \in E$ if $I_n'(z) = 0$ then

$$(2.11)\qquad \frac{3\mu\sigma+2}{8}\int_0^{2\pi}(H_n(z)+\beta_0)dt + \beta_5 \leq \left(\left(\frac{1}{2}A(z)\right)^2+1\right)^{1/2} + \beta_6 .$$

Proof. From $\langle I_n'(z),z\rangle = 0$, we get that

$$\frac{1}{2}A(z) = \frac{1}{2}\int_0^{2\pi}(H_n'(z)\cdot z+\beta_4)dt + \frac{1}{2}\int_0^{2\pi} f\cdot z\,dt - \pi\beta_4$$

$$(2.12)\qquad \geq \frac{\mu\sigma}{2}\int_0^{2\pi}(H_n(z)+\beta_0)dt + \frac{1}{2}\int_0^{2\pi} f\cdot z\,dt - \pi\beta_4 \quad \text{(by (2.6))}$$

$$\geq \frac{3\mu\sigma+2}{8}\int_0^{2\pi}(H_n(z)+\beta_0)dt + \frac{\mu\sigma-2}{16}\alpha_0\|z\|^{\mu\sigma}_{L^{\mu\sigma}}$$

$$- \|f\|_{L^2}\|z\|_{L^2} - \pi\beta_4 - 1 .$$

Since $\mu\sigma > 2$, (2.11) holds.

□

Let $\chi \in C^{\infty}(\mathbf{R},\mathbf{R})$ such that $\chi(s) = 1$ if $s \leq 1$, $\chi(s) = 0$ if $s \geq \tau$, and $-\frac{2}{\tau-1} < \chi'(s) < 0$ if $1 < s < \tau$, where τ is defined before Lemma 2.8.

For $n \in \mathbf{N}$ and $z \in E$ we define

$$\varphi_0(z) = ((\tfrac{1}{2} A(z))^2 + 1)^{1/2} + \beta_6 ,$$

$$\varphi(z) = \frac{3\mu\sigma+2}{8} \int_0^{2\pi} (H(z)+\beta_0)dt + \beta_5 , \quad \varphi_n(z) = \frac{3\mu\sigma+2}{8} \int_0^{2\pi} (H_n(z)+\beta_0)dt + \beta_5 ,$$

$$\psi(z) = \chi\left(\frac{\varphi(z)}{\tau\varphi_0(z)}\right) , \quad \psi_n(z) = \chi\left(\frac{\varphi_n(z)}{\tau\varphi_0(z)}\right) ,$$

$$J(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H(z)dt - \psi(z) \int_0^{2\pi} f\cdot z \, dt ,$$

$$J_n(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H_n(z)dt - \psi_n(z) \int_0^{2\pi} f\cdot z \, dt .$$

For these functionals on E we have

Lemma 2.13. 1°. $\psi_n \in C^1(E,\mathbf{R})$, $J_n \in C^1(E,\mathbf{R})$, $\forall\, n \in \mathbf{N}$.

2°. $\psi \in C(E,\mathbf{R})$, $J \in C(E,\mathbf{R})$.

Proof. For 1° we refer to [5,17], 2° follows from (H3) and Lemma 2.3.

□

Lemma 2.14. For any $m, n \in \mathbf{N}$, $m > n$ and $z \in E$,

1°. $0 \leq \frac{1}{2} \int_0^{2\pi} (H_m(z)-H_n(z))dt \leq J_n(z) - J_m(z) \leq \frac{3}{2} \int_0^{2\pi}(H_m(z)-H_n(z))dt$,

2°. $0 \leq \frac{1}{2} \int_0^{2\pi} (H(z)-H_n(z))dt \leq J_n(z) - J(z) \leq \frac{3}{2} \int_0^{2\pi} (H(z)-H_n(z))dt$.

Proof. We only prove 1°. The proof of 2° is similar. Let $\operatorname{supp} \psi_n$ be the closure of $\{z \in E \mid \psi_n(z) \neq 0\}$ in E.

If $z \notin \operatorname{supp} \psi_n \cup \operatorname{supp} \psi_m$, then by 4° of Proposition 2.5

$$J_n(z) - J_m(z) = \int_0^{2\pi} (H_m(z)-H_n(z))dt \geq 0 .$$

if $z \in \text{supp}\,\psi_n \cup \text{supp}\,\psi_m$, then $\varphi_j(z) < \tau^2\varphi_0(z)$ for $j = n$ or $j = m$.

By Lemma 2.8,

$$\varphi_j(z) > \frac{\tau(3\mu\sigma+2)}{2(\tau-1)} \|f\|_{L^2}\|z\|_{L^2} .$$

So

$$(2.15) \qquad \frac{\frac{2}{\tau-1} \cdot \frac{3\mu\sigma+2}{8} \|f\|_{L^2}\|z\|_{L^2}}{\tau\varphi_0(z)} < \frac{\varphi_j(z)}{2\tau^2\varphi_0(z)} < \frac{1}{2} .$$

On the other hand

$$J_n(z) - J_m(z) = \int_0^{2\pi} (H_m(z)-H_n(z))dt + \chi'(\zeta) \frac{\varphi_m(z)-\varphi_n(z)}{\tau\varphi_0(z)} \int_0^{2\pi} f\cdot z\, dt ,$$

where we have used the mean value theorem with a number ζ between $\frac{\varphi_m(z)}{\tau\varphi_0(z)}$ and $\frac{\varphi_n(z)}{\tau\varphi_0(z)}$. By the definition of φ_m and φ_n, we get,

$$(2.16) \quad J_n(z) - J_m(z) = \left(1 + \frac{\chi'(\zeta)\frac{3\mu\sigma+2}{8}\int_0^{2\pi} f\cdot z\, dt}{\tau\varphi_0(z)}\right) \int_0^{2\pi}(H_m(z)-H_n(z))dt .$$

Since $|\chi'(\xi)| < \frac{2}{\tau-1}$, $\forall\ \xi \in \mathbf{R}$, by (2.15) we get that

$$\frac{1}{2} < 1 + \frac{\chi'(\zeta)\frac{3\mu\sigma+2}{8}\int_0^{2\pi} f\cdot z\, dt}{\tau\varphi_0(z)} < \frac{3}{2} .$$

Combining with 4° of Proposition 2.5 and (2.16) we get the proof of 1°.

□

<u>Corollary 2.17</u>. For any $n \in \mathbf{N}$ and $z \in E$

$$J_n(z) > J_{n+1}(z) > J(z) .$$

<u>Lemma 2.18</u>. There exists $\beta_7 > 0$ independent of n such that for any $n \in \mathbf{N}$, $z \in E$ and $M > \beta_8$, if $J_n(z) > M$ then $\varphi_0(z) > \frac{M}{2}$.

<u>Proof</u>. Since $\mu\sigma > 2$ and

$$J_n(z) < \frac{1}{2} A(z) - \alpha_0\|z\|^{\mu\sigma}_{L^{\mu\sigma}} + \|f\|_{L^2}\|z\|_{L^2} + 2\pi\beta_0 ,$$

there exists $C > 0$ independent of n such that

$$\varphi_0(z) = ((\tfrac{1}{2} A(z))^2 + 1)^{1/2} + \beta_6 \geq J_n(z) + \beta_6 - C$$

and this yields the Lemma.

□

Lemma 2.19. There exists a constant $\beta_8 > 0$ independent of n such that for any $n \in \mathbf{N}$ and $z \in E$, if $J_n(z) \geq \beta_8$ and $\langle J_n'(z), z\rangle = 0$, then $J_n(z) = I_n(z)$ and $J_n'(z) = I_n'(z)$.

Proof. For any $z, \zeta \in E$, we have that

$$(2.20)\qquad \begin{aligned}\langle J_n'(z), \zeta\rangle &= (1 + T_{n,1}(z))\overline{A}(z,\zeta) \\ &\quad - (1 + T_{n,2}(z)) \int_0^{2\pi} H_n'(z)\cdot\zeta \, dt - \psi_n(z) \int_0^{2\pi} f\cdot\zeta \, dt\end{aligned}$$

where

$$\overline{A}(z,\zeta) = \frac{1}{2} \int_0^{2\pi} (\dot{z}\cdot \zeta + \dot{\zeta}\cdot Jz)dt ,$$

$$T_{n,1}(z) = \chi'\left(\frac{\varphi_n(z)}{\tau\varphi_0(z)}\right) \frac{A(z)\varphi_n(z)}{2\tau\varphi_0^2(z)(\varphi_0(z)-\beta_6)} \int_0^{2\pi} f\cdot z \, dt$$

$$T_{n,2}(z) = \chi'\left(\frac{\varphi_n(z)}{\tau\varphi_0(z)}\right) \frac{3\mu\sigma+2}{8\tau\varphi_0(z)} \int_0^{2\pi} f\cdot z \, dt .$$

If for large enough β_8, we have

$$(2.21)\qquad |T_{n,1}(z)| < \frac{\mu\sigma-2}{16\mu\sigma} , \quad |T_{n,2}(z)| < \frac{\mu\sigma-2}{16\mu\sigma} .$$

Then from (2.20) with $\zeta = z$, we get

$$\frac{1}{2} A(z) \geq \frac{3\mu\sigma+2}{8} \int_0^{2\pi} (H_n(z)+\beta_0)dt + \frac{\mu\sigma-2}{16} \alpha_0 \|z\|^{\mu\sigma}_{L^{\mu\sigma}} - \|f\|_{L^2} \|z\|_{L^2} - \beta_4 - 1 .$$

This is (2.12). So $\varphi_0(z) \geq \varphi_n(z)$, thus $\psi_n(z) = 1$ and $\psi_n'(z) = 0$. This yields the lemma. Therefore we reduce to the proof of (2.21).

If $z \notin \operatorname{supp} \psi_n$, $T_{n,1}(z) = T_{n,2}(z) = 0$. If $z \in \operatorname{supp} \psi_n$, then

$$\tau^2 \varphi_0(z) \geq \varphi_n(z) \geq \frac{3\mu\sigma+2}{8} \alpha_0 \|z\|^{\mu\sigma}_{L^{\mu\sigma}} .$$

Thus

$$\left|T_{n,1}(z)\right| \leq \frac{\tau}{\tau-1} \|f\|_{L^2} (2\pi)^{\frac{\mu\sigma-2}{2\mu\sigma}} \frac{\|z\|_{L^{\mu\sigma}}}{\varphi_0(z)} \leq M(\varphi_0(z))^{\frac{1}{\mu\sigma}-1} ,$$

Similarly

$$\left|T_{n,2}(z)\right| \leq M(\varphi_0(z))^{\frac{1}{\mu\sigma}-1}$$

for some constant $M > 0$ independent of n and z, by Lemma 2.18, this implies (2.21) and completes the proof of the Lemma.

□

We say J_n satisfies the Palais-Smale condition (PS) if whenever a sequence $\{z_j\}$ in E satisfies that $\{J_n(z_j)\}$ is bounded and $J_n'(z_j) \to 0$ as $j \to +\infty$, then $\{z_j\}$ possesses a convergent subsequence.

Lemma 2.22. For any $n \in \mathbb{N}$, J_n satisfies (PS) on $[J_n]_{\beta_8} \equiv \{z \in E \mid J_n(z) \geq \beta_8\}$, where the constant $\beta_8 > 0$ is defined in Lemma 2.19 which is independent of n.

Proof. Since $J_n'(z_j) \to 0$, we may assume $\left|\langle J_n'(z_j), z\rangle\right| \leq \|z\|_E$ $\forall z \in E$. By the assumption $\beta_8 \leq J_n(z_j) \leq M$, from (2.20) and (2.21) we get

$$M + \|z_j\|_E \geq J_n(z_j) - \frac{1}{2(1+T_{n,1}(z_j))} \langle J_n'(z_j), z_j\rangle$$

$$= \frac{1+T_{n,2}(z_j)}{2(1+T_{n,1}(z_j))} \int_0^{2\pi} H_n'(z_j)\cdot z_j dt - \int_0^{2\pi} H_n(z_j)dt$$

$$- \left(1 - \frac{1}{2(1+T_{n,1}(z_j))}\right) \psi_n(z_j) \int_0^{2\pi} f\cdot z_j dt$$

$$\geq \frac{\mu\sigma-2}{16\mu\sigma} \int_0^{2\pi} H_n'(z_j)\cdot z_j dt + \frac{1}{4} (\mu\sigma-2) \int_0^{2\pi}(H_n(z_j)+\beta_0)dt$$

$$- \frac{2}{3} \|f\|_{L^2}\|z_j\|_{L^2} - 2\pi(\beta_4 - \beta_0) .$$

Therefore there is $M_1 > 0$ independent of n and j such that

$$(2.23) \qquad \|z_j\|_{L^{\mu\sigma}}^{\mu\sigma} + \int_0^{2\pi}(H_n'(z_j)\cdot z_j + \beta_4)dt \le M_1(\|z_j\|_E + 1) \quad .$$

Write $z_j = z_j^+ + z_j^- + z_j^0$. Since $z_j^0 = \frac{1}{2\pi}\int_0^{2\pi} z_j dt$, there is $M_2 > 0$ independent of n, j such that

$$(2.24) \qquad |z_j^0| \le \|z_j\|_{L^{\mu\sigma}} \le M_2(\|z_j\|_E^{\frac{1}{\mu\sigma}} + 1) \quad .$$

From (2.20) for $\langle J_n'(z_j), z_j^+\rangle$, we get

$$\frac{1}{2}\|z_j^+\|_E^2 \le (1+T_{n,1}(z_j))A(z_j^+) \le \frac{3}{2}\int_0^{2\pi}|H_n'(z_j)|\ |z_j^+|dt + \|f\|_{L^2}\|z_j^+\|_{L^2} + \|z_j^+\|_E \quad .$$

By 6° of Proposition 2.5 and (2.23) we get

$$\int_0^{2\pi}|H_n'(z_j)|\ |z_j^+|dt \le \left(\int_0^{2\pi}|H_n'(z_j)|^{\lambda_0}dt\right)^{1/\lambda_0}\|z_j^+\|_{L^{\lambda_0/(\lambda_0-1)}}$$

$$\le M_3(n)\left(\|z_j\|_E^{1+1/\lambda_0} + 1\right)$$

for some constant $M_3(n) > 0$ independent of j. Thus there is $M_4(n) > 0$ independent of j such that

$$\|z_j^+\|_E^2 \le M_4(n)(\|z_j\|_E^{1+1/\lambda_0} + 1) \quad .$$

Similarly

$$\|z_j^-\|_E^2 \le M_5(n)(\|z_j\|_E^{1+1/\lambda_0} + 1)$$

for some $M_5(n) > 0$ independent of J. Combining with (2.24), we get a constant $M_6(n) > 0$ independent of j such that

$$(2.25) \qquad \|z_j\|_E \le M_6(n) \quad .$$

Let $P^\pm : E \to E^\pm$ be the orthogonal projections. From (2.20)

$$P^\pm J_n'(z_j) = \pm(1 + T_{n,1}(z_j))z_j^\pm \pm \mathcal{P}_n(z_j)$$

where $\mathcal{P}_n$ is a compact operator by (H1), (H2) and Proposition 2.5. Since

$$|T_{n,1}(z_j)| < \frac{1}{16},$$

$$\pm z_j^{\pm} = (1 + T_{n,1}(z_j))^{-1}P^{\pm}J_n'(z_j) - (1 + T_{n,1}(z_j))^{-1}P^{\pm}\mathcal{P}_n(z_j) \quad .$$

By (2.25) this shows that $\{z_j^+\}$ and $\{z_j^-\}$ are precompact in E. By (2.25) $\{z_j^0\}$ is also precompact, therefore $\{z_j\}$ is precompact in E, and the proof is complete.

□

<u>Lemma 2.26:</u> There exists a constant $\beta_9 > 0$ such that

(2.27) $$|J(z) - J(T_\theta z)| \le \beta_9(\log^{1/q_1}(|J(z)| + 1) + 1) \quad \forall z \in E \ .$$

where q_1 is defined in (H3).

<u>Proof:</u> A direct application of Hölder inequality shows that there exists $c_1 > 0$ such that

$$e^{\tau_1\left(\frac{1}{2\pi}\int_0^{2\pi}|z|dt\right)^{q_1}} \le \frac{1}{\pi}\int_0^{2\pi} e^{\tau_1|z|^{q_1}} dt + c_1 \quad \forall z \in E \ .$$

If $z \in \text{supp}\,\psi$, $|J(z)| \ge \left(\frac{3\mu\sigma+2}{8\tau^2} - 1\right)\int_0^{2\pi}(H(z) + \beta_0)dt - c_2\int_0^{2\pi}|z|dt - c_3$

$$\ge c_4\int_0^{2\pi} e^{\tau_1|z|^{q_1}} dt - c_5 \ .$$

Here we used (H3), and c_j's denote positive constants. Therefore

$$|J(z) - J(T_\theta z)| \le 2\,\psi(z)\int_0^{2\pi}|f\cdot z|dt \le c_6\psi(z)\int_0^{2\pi}|z|$$

$$\le \beta_9(\log^{1/q_1}(|J(z)| + 1) + 1)$$

for some $\beta_9 > 0$ and the proof is complete.

□

§3. A minimax structure

For $k \in \mathbb{N}$, $k > N+1$, we define $V_k(E) = E^+_{N+1,k} \oplus E^- \oplus E^0$. By (2.6) and Corollary 2.17, there exists R_k for $k > N+1$ such that $1 < R_k < R_{k+1}$ and $J(z) < J_n(z) < 0$ $\forall\, n \in \mathbb{N}$, $z \in V_k(E)$ with $\|z\|_E > R_k$.

Let $D_k(E) = V_k(E) \cap B_k(E)$, $B_k(E) = \{z \in E \mid \|z\|_E < R_k\}$.

For $z = z^0 + z^+ + z^- \in E$ we write

$$(3.1)\quad z^+ = \sum_{k>N+1} \rho_k e^{i\hat{\alpha}_k} \hat{\varphi}_k,\ z^- = \sum_{k>N+1} \sigma_k e^{i\hat{\beta}_k} \hat{\psi}_k \ \text{ where } \rho_k, \sigma_k > 0,\ \hat{\alpha}_k, \hat{\beta}_k \in [0,2\pi]\ .$$

Then we have

$$\|z\|_E^2 = |z^0|^2 + 2\pi \sum_{k>N+1} \bar{k}(\rho_k^2 + \sigma_k^2), \text{ and}$$

$$T_\theta z = z^0 + \sum_{k>N+1} (\rho_k e^{i(\hat{\alpha}_k + \bar{k}\theta)} \hat{\varphi}_k + \sigma_k e^{i(\hat{\beta}_k + \bar{k}\theta)} \hat{\psi}_k) \quad \forall\, \theta \in [0,2\pi]\ .$$

We define a new S^1-action on E by

$$\hat{T}_\theta z = z^0 + \sum_{k>N+1} (\rho_k e^{i(\hat{\alpha}_k + \theta)} \hat{\varphi}_k + \sigma_k e^{i(\hat{\beta}_k + \theta)} \hat{\psi}_k) \quad \forall\, \theta \in [0,2\pi]$$

for z with expression (3.1) and denote E with $\hat{T}_\theta$ by X. We define an S^1-action on $X \times E$ by

$$\tilde{T}_\theta(x,z) = (\hat{T}_\theta x, T_\theta z) \quad \forall\, (x,z) \in X \times E \text{ and } \theta \in [0,2\pi]\ .$$

We also define S^1-invariant sets, equivariant maps, invariant functionals for X, E, $X \times E$ in a usual way (cf. [9]). Let $\mathcal{E}$ (or $\mathcal{X}, \mathcal{F}$) denote the family of closed (in E) (or $X, X\times E$) S^1-invariant subsets in $E\backslash\{0\}$ (or $X\backslash\{0\}$, $X \times E\backslash\{0\}$). Then $\mathcal{F}$ contains $\times \{0\}$ and $\{0\} \times \mathcal{E}$. For $B \in \mathcal{X}$, we say a map $h : B \to E$ is S^1-equivariant if $h(\hat{T}_\theta x) = T_\theta h(x)$ $\forall\, x \in B$ and $\theta \in [0,2\pi]$. We also denote by $V_k(X)$, $D_k(X)$, $B_k(X)$ the sets in X corresponding to $V_k(E)$, $D_k(E)$, $B_k(E)$, etc. We introduce the Fadell-

Rabinowitz cohomological index theory on F (cf. [9]).

Lemma 3.2: There is an index theory on F i.e. a mapping $\gamma : F \to \{0\} \cup \mathbb{N} \cup \{+\infty\}$ such that if $A,B \in F$

1°. $\gamma(A) \leq \gamma(B)$. if there exists $h \in C(A,B)$ with h being S^1-equivariant.

2°. $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

3°. If $B \subset (X\times E)\backslash(X^0\times E^0)$ and B is compact, then $\gamma(B) < \infty$ and there is a constant $\delta > 0$ such that $\gamma(N_\delta(B,X\times E)) = \gamma(B)$ where $N_\delta(B,X\times E) = \{z\in X\times E \mid \|z-B\|_{X\times E} < \delta\}$.

4°. If $S \subset (X\times E)\backslash(X^0\times E^0)$ is a $2n-1$ dimensional invariant sphere, then $\gamma(S) = n$.

By identifying X with $X \times \{0\}$, and E with $\{0\} \times E$, we may view that the index theory γ is defined on both X and E.

For $x \in X$, $z \in E$, we write $z \sim x$ if $z^0 = x^0$, $\rho_k(z) = \rho_k(x)$, $\sigma_k(z) = \sigma_k(x)$, $\hat{\alpha}_k(z) = \hat{\alpha}_k(x)$ and $\hat{\beta}_k(z) = \hat{\beta}_k(x)$ $\forall\ k \geq N+1$.

For $x = x^0 + \sum_{k\geq N+1} \left(\rho_k e^{i\hat{\alpha}_k} \hat{\varphi}_k + \sigma_k e^{i\hat{\beta}_k} \hat{\psi}_k\right) \in X$, we define

$$h(x) = x^0 + \sum_{k\geq N+1} \left(\rho_k e^{i\overline{k}\hat{\alpha}_k} \hat{\varphi}_k + \sigma_k e^{i\overline{k}\hat{\beta}_k} \hat{\psi}_k\right) .$$

About this map h, we have

Lemma 3.3: 1°. $h \in C(X,E)$ and is surjective.

2°. h is S^1-equivariant.

3°. $h(\partial B_\rho(X) \cap V_k(X)) = \partial B_\rho(E) \cap V_k(E)$ $\forall\ \rho > 0,\ k \geq N+1$.

4°. $h(x) = z$ if $x \in X^0$ and $x \sim z$.

5°. If $\{h(x_n)\}$ is convergent in E, $\{x_n\}$ is precompact in X.

Proof: $1° \sim 4°$ are direct consequences of the definition of h. Suppose $\{x_n\} \subset X$ and $h(x_n) \to z$ in E as $n \to \infty$. Write $x_n = (x_n^0; \rho_k(n), \alpha_k(n); \sigma_k(n); \beta_k(n))$. Since $2\pi\bar{k}(\rho_k(n) - \rho_k(m))^2 \le \|h(x_n) - h(x_m)\|_E^2$, we get that $\{\rho_k(n)\}$, similarly $\{\sigma_k(n)\}$, is convergent for each $k \in \mathbb{N}$. Since $\{x_n^0\}$ is precompact, and $\alpha_k(n), \beta_k(n) \in [0,2\pi]$, we can choose a sequence $\{n_j\}$ in $\mathbb{N}$ such that $\{x_{n_j}^0\}$, $\{\alpha_k(n_j)\}$ and $\{\beta_k(n_j)\}$ are convergent for every fixed $k \in \mathbb{N}$. Then $\{x_{n_j}\}$ is convergent in X. This proves $5°$.

□

We name the above map $h : X \to E$ by "id". With the aid of "id" we can define minimax structures now.

Definition 3.4. For $j \in \mathbb{N}$, $j > N+1$, define Γ_j to be the family of such maps h, which saisfy the following conditions:

1°. $h \in C(D_j(X),E)$ and is S^1-equivariant.

2°. $h = id$ on $(\partial B_j(X) \cap V_j(X)) \cup (X^0 \cap D_j(X)) \equiv F_j(X)$.

3°. $P^- h(x) = \alpha(x)P^- id(x) + \beta(x)$, $\forall\, x \in D_j(X)$, where $\alpha \in C(D_j(X), [1,\bar{\alpha}])$ with $1 < \bar{\alpha} < +\infty$ depending on h and is S^1-invariant, $\beta \in C(D_j(X), E^-)$ is compact and S^1-equivariant, and $\beta = 0$ on $F_j(X)$.

4°. $h(D_j(X))$ is bounded in E.

Definition 3.5. For $j \in \mathbb{N}$, $j > N+1$, define Λ_j to be the family of maps h, which satisfy

1°. $h \in C(D_{j+1}(X), E)$ and $h|_{D_j(X)} \in \Gamma_j$.

2°. $h = id$ on $(\partial B_{j+1}(X) \cap V_{j+1}(X)) \cup ((B_{j+1}(X) \backslash B_j(X)) \cap V_j(X)) = G_j(X)$.

3°. $P^- h(x) = \alpha(x)P^- id(x) + \beta(x)$, for any $x \in D_{j+1}(X)$ where $\alpha \in C(D_{j+1}(X), [1,\tilde{\alpha}])$ with $1 < \tilde{\alpha} < +\infty$ depending on h, $\beta \in C(D_{j+1}(X), E^-)$ is compact and $\beta = 0$ on $G_j(X)$, and α, β

are extensions of the corresponding maps defined in 1° via the definition of Γ_j.

4°. $h(D_{j+1}(X))$ is bounded in E.

Remark. $id \in \Gamma_j \cap \Lambda_j$ for any $j \geq N+1$.

Lemma 3.6. For $j \geq N+1$, any $h \in \Gamma_j$ can be extended to a map in Λ_j.

Proof. For $h \in \Gamma_j$, define $h = id$ on $G_j(X)$. This also extends α and β in 3° of Definition 3.4 of h by $\alpha = 1$ and $\beta = 0$ on $G_j(X)$. Now we use Dugundji extension theorem [7] to extend α, β to the whole $D_{j+1}(X)$. Since by this theorem the image of the extension mapping is contained in the closed convex hull of the original image, $\alpha \in C(D_{j+1}(X), [1,\bar{\alpha}])$ and $\beta \in C(D_{j+1}(X), E^-)$ is compact and 4° of Definition 3.5 holds. We use this theorem again to extend P^+h, P^0h to the whole $D_{j+1}(X)$. ($P^0 : E \to E^0$ is the orthogonal projector.) Then define $h = P^+h + P^-h + P^0h$. It is easy to check that $h \in \Lambda_j$.

□

Now for $k \in \mathbb{N}$, $k \geq N+1$ we define

$$A_k = \{\overline{h(D_j(X)\setminus Y)} \mid j \geq k,\ h \in \Gamma_j,\ Y \in \mathcal{X} \text{ with } \gamma(Y) \leq j-k\} ,$$

$$\mathcal{B}_k = \{\overline{h(D_{j+1}(X)\setminus Y)} \mid j \geq k,\ h \in \Lambda_j,\ Y \in \mathcal{X} \text{ with } \gamma(Y) \leq j-k\} ,$$

$$a_k = \inf_{A\in A_k} \sup_{z\in A} J(z) , \quad b_k = \inf_{B\in \mathcal{B}_k} \sup_{z\in B} J(z)$$

$$a_k(n) = \inf_{A\in A_k} \sup_{z\in A} J_n(z) , \quad b_k(n) = \inf_{B\in \mathcal{B}_k} \sup_{z\in B} J_n(z) .$$

Since $id \in \Gamma_j \cap \Lambda_j$ and $0 \in A \cap B\ \forall A \in A_j, B \in \mathcal{B}_\ell$, $-\infty < a_k, b_k, a_k(n), b_k(n) < +\infty$.

§4. Properties of sequences of minimax values.

We have

Lemma 4.1. 1°. $\{a_k\}$ and $\{b_k\}$ are increasing sequences.

2°. $\{a_k(n)\}$ and $\{b_k(n)\}$ are increasing sequences for fixed $n \in \mathbf{N}$.

3°. $a_k \leq b_k$, $a_k(n) \leq b_k(n)$ $\forall\, n, k \in \mathbf{N}$.

4°. $a_k \leq a_k(n+1) \leq a_k(n)$, $b_k \leq b_k(n+1) \leq b_k(n)$ $\forall\, n, k \in \mathbf{N}$.

Proof. 1° and 2° are due to the fact that $\mathcal{A}_{k+1} \subseteq \mathcal{A}_k$ and $\mathcal{B}_{k+1} \subseteq \mathcal{B}_k$. For any $B = \overline{h(D_{j+1}(X)\backslash Y)} \in \mathcal{B}_k$, let $A = \overline{h(D_j(X)\backslash Y)}$, then $A \subset B$ and $A \in \mathcal{A}_k$, so 3° holds. 4° follows from Corollary 2.17.

□

Lemmar 4.2. For any fixed $k \in \mathbf{N}$, $k \geq N+1$.

1°. $\lim\limits_{n\to\infty} a_k(n) = a_k$

2°. $\lim\limits_{n\to\infty} b_k(n) = b_k$.

Proof. We only prove 1°. 2° can be done similarly.

Given any $\varepsilon > 0$, by the definition of a_k, there is $A_0 \in \mathcal{A}_k$ such that $\sup\limits_{z\in A_0} J(z) \leq a_k + \frac{\varepsilon}{3}$. So

$$(4.3) \qquad a_k(n) \leq \sup_{z\in A_0} J_n(z) \leq a_k + \frac{\varepsilon}{2} + \sup_{z\in A_0} D_n(z) \quad \forall\, n \in \mathbf{N}$$

where $D_n(z) = J_n(z) - J(z)$. Since A_0 is bounded in E, by Lemma 2.3, $\sup\limits_{z\in A_0} D_n(z) < +\infty$. Thus for every $n \in \mathbf{N}$, there exists $z_n \in A_0$ such that

$$(4.4) \qquad \sup_{z\in A_0} D_n(z) \leq D_n(z_n) + \frac{\varepsilon}{3} \; .$$

Since E is compactly imbedded into $L^1(S^1,\mathbf{R}^{2N})$, $\{z_n\}$ possesses a

subsequence $\{z_{n_j}\}$ which converges to some z_0 in $L^1(S^1,\mathbb{R}^{2N})$. From real analysis (cf. [14]) $\{z_{n_j}\}$ has a subsequence which converges to z_0 almost everywhere. We still denote it by $\{z_{n_j}\}$.

Let $Q = \{t \in [0,2\pi] \mid |z_0(t)| < \infty\} \cap \{t \in [0,2\pi] \mid z_{n_j}(t) \to z_0(t)$ as $j \to \infty\}$, then Q has Lebesgue measure 2π. For any $t \in Q$, there is $N_1(t) > 0$ such that $|z_{n_j}(t)| < |z_0(t)| + 1$, $\forall\ j \geq N_1(t)$. Choose $N_2(t) \geq N_1(t)$ such that $K_{n_j} \geq |z_0(t)| + 1$, $\forall\ j \geq N_2(t)$ where $\{K_n\}$ is defined in Proposition 2.5. Then $H_{n_j}(z_{n_j}(t)) = H(z_{n_j}(t))$, $\forall\ j \geq N_2(t)$. This shows that $H(z_{n_j}) - H_{n_j}(z_{n_j}) \to 0$ almost everywhere as $j \to \infty$. Thus $H(z_{n_j}) - H_{n_j}(z_{n_j}) \to 0$ in measure as $j \to \infty$.

Since $\{z_{n_j}\}$ are bounded in E, by (H3) and Lemma 2.3, $\{H(z_{n_j}) - H_{n_j}(z_{n_j}) \mid j \in \mathbb{N}\}$ are bounded in $L^2(S^1,\mathbb{R}^{2N})$. So by a theorem of De La Vallée-Poussin (Theorem VI.3.7 [14]) with $\Phi(u) = u$, $\{H(z_{n_j}) - H_{n_j}(z_{n_j}) \mid j \in \mathbb{N}\}$ have equi-absolute continuous integrals.

Now we can apply D. Vitali's theorem (Theorem VI.3.2 [14]) and get a constant $N_3 > 0$ such that

$$\left|\int_0^{2\pi}(H(z_{n_j}) - H_{n_j}(z_{n_j}))dt\right| < \frac{2\varepsilon}{9} \quad , \quad \forall\ j \geq N_3 \quad .$$

Combining with (4.4) and Lemma 2.14 we get

$$0 \leq D_{n_j}(z_{n_j}) < \frac{\varepsilon}{3} \qquad \forall\ j \geq N_3 \quad .$$

Let $N_0 = n_{N_3}$, then combining with (4.3) yields $a_k(N_0) \leq a_k + \varepsilon$. By Lemma 4.1 we get

$$a_k \leq a_k(n) \leq a_k(N_0) \leq a_k + \varepsilon \qquad \forall\ n \geq N_0 \quad .$$

This completes the proof.

□

§5. An upper estimate for the growth rate of $\{a_k\}$

By (2.27) we get

(5.1) $J(T_\theta z) \le J(z) + \beta_9(\log^{1/q_1}(|J(z)| + 1) + 1) \quad \forall z \in E$ and $\theta \in [0,2\pi]$

So there is $M_1 > 0$ depending only on β_9 and q_1 such that

(5.2) $$J(z) + \beta_9(\log^{1/q_1}(|J(z)| + 1) + 1) \le 0 \quad \text{if} \quad J(z) \le -M_1 .$$

We shall prove the following claim in §6: $a_k \to +\infty$ as $k \to \infty$. So there is a $k_0 \in \mathbf{N}$, $k_0 \ge N+1$ such that

(5.3) $$a_k \ge M_1+1, \quad \forall k \ge k_0 .$$

Proposition 5.4. Assume that there is $k_1 \ge k_0$ such that $b_k = a_k$, $\forall k \ge k_1$. Then there is $M = M(k_1) > 0$ such that

(5.5) $$a_k \le Mk (\log k)^{1/q_1} \quad \forall k \ge k_1 .$$

Proof. Assuming the following inequality for a moment

(5.6) $$\inf_{A\in\mathcal{A}_{k+1}} \sup_{z\in A}\Big(\max_{\theta\in[0,2\pi]} J(T_\theta z)\Big) \le \inf_{B\in\mathcal{B}_k} \sup_{z\in B}\Big(\max_{\theta\in[0,2\pi]} J(T_\theta z)\Big) .$$

For $k \ge k_1$ we get

$$a_{k+1} \le \inf_{B\in\mathcal{B}_k} \sup_{z\in B}\Big(\max_{\theta\in[0,2\pi]} J(T_\theta z)\Big) .$$

For any $\varepsilon > 0$, by the definition of b_k, there is a $B \in \mathcal{B}_k$ such that $\sup_{z\in B} J(z) \le b_k + \varepsilon = a_k + \varepsilon$. For this B, using (5.1), (5.2), (5.3) we get that

$$J(T_\theta z) \le a_k + \varepsilon + \beta_9(\log^{1/q_1}(a_k + \varepsilon + 1) + 1) \quad \forall z \in B \text{ and } \theta \in [0,2\pi] .$$

Therefore there is $M_2 > 0$ depending only on β_9 and q_1 such that

$$a_{k+1} \le a_k + \varepsilon + M_2(\log^{1/q_1}(a_k + \varepsilon) + 1) .$$

Letting $\varepsilon \to 0$ yields

(5.7) $$a_{k+1} \le a_k + M_2(\log^{1/q_1} a_k + 1) .$$

Write $\delta_k = a_k(k \log^p k)^{-1}$, $p = 1/q_1$. We need to prove $\{\delta_k\}$ is bounded. Now

(5.7) becomes

$$(5.8)\quad \delta_{k+1}(k+1)\log^p(k+1) \le \delta_k\, k\log^p k + M_2((\log \delta_k + \log(k\log^p k))^p + 1) .$$

If $\delta_{k+1} > \delta_k$ we get

$$\delta_k \log^{-p}\delta_k \le M_2 \frac{\left(1+\frac{\log(k\log^p k)}{\log \delta_k}\right)^p + \frac{1}{\log \delta_k}}{k(\log^p(k+1)-\log^p k)+\log^p(k+1)}$$

If $\delta_k > e$, then

$$\delta_k \log^{-p}\delta_k \le M_2 \frac{(1+\log k + p\log\log k)^p + 1}{\log^p(k+1)} .$$

Thus there is $M_3 > 0$ depending only on P and M_2 such that $\delta_k \le M_3$. Then from (5.8) it is easy to see that there is a constant $M_4 > 0$ depending only on M_3 and p such that $\delta_{k+1} \le M_4$.

Therefore $\delta_{k+1} \le \max\{\delta_k, M_4\}$, $\forall\, k \ge k_1$. So $\delta_k \le \max\{\delta_{k_1}, M_4\}$, $\forall\, k \ge k_1$. Let $M = \max\{\delta_{k_1}, M_4\}$. This yields (5.5). Therefore we reduce to the proof of (5.6), i.e.

<u>Lemma 5.9</u>. If L is a continuous S^1-invariant functional on E, then

$$(5.10)\qquad \inf_{A\in A_{k+1}} \sup_{z\in A} L(z) \le \inf_{b\in B_k} \sup_{z\in B} L(z) \qquad \forall\, k \ge N+1 .$$

<u>Proof</u>. Given any $B \in B_k$, by the definition, there are $j \ge k$, $h_1 \in \Lambda_j$, $Y \in X$ with $\gamma(Y) \le j-k$ such that $B = \overline{h_1(D_{j+1}(X)\setminus Y)}$. Let

$$U_j(X) = \{x \in D_{j+1}(X) \mid x = x' + \rho_{j+1}\hat{\varphi}_{j+1},\ x' \in V_j(X),\ \rho_{j+1} \ge 0 \text{ and } \|x\|_E \le R_{j+1}\} .$$

By the definition of h_1, for any $x \in U_j(X)$, $P^- h_1(x) = \alpha_1(x)P^- id(x) + \beta_1(x)$, where α_1 and β_1 are given by 3° of Definition 3.4. We define

$$\alpha(x) = \alpha_1(x), \quad \beta(x) = \beta_1(x) \qquad \forall\, x \in U_j(x)$$

$$\alpha(\hat{T}_\theta x) = \alpha_1(x),\ \beta(\hat{T}_\theta x) = T_\theta\beta(x) \qquad \forall\, x \in U_j(X) \text{ and } \theta \in [0,2\pi) .$$

Note that $D_{j+1}(X) = \bigcup_{\theta\in[0,2\pi]} \hat{T}_\theta U_j(X)$ and for any given $y \in D_{j+1}(X)$. $\hat{T}_\theta x = y$ possesses a unique solution (x,θ) in $U_j(X) \times [0,2\pi)$. So α and β are well defined, $\alpha \in C(D_{j+1}(X), [1,\bar{\alpha}_1])$ is S^1-invariant, $\beta \in C(D_{j+1}(X), E^-)$ is compact and S^1-equivariant.

Define

$$h^-(x) = \alpha(x)P^- id(x) + \beta(x) \qquad \forall\, x \in D_{j+1}(X)$$

$$h^+(x) = P^+h_1(x),\; h^0(x) = P^0h_1(x) \qquad \forall\, x \in U_j(x)$$

$$h^+(\hat{T}_\theta x) = T_\theta h^+(x),\; h^0(\hat{T}_\theta x) = h^0(x) \qquad \forall\, x \in U_j(x) \text{ and } \theta \in [0,2\pi)$$

and

$$h(x) = h^+(x) + h^-(x) + h^0(x) \qquad \forall\, x \in D_{j+1}(X).$$

Then $h \in \Gamma_{j+1}$, $h = h_1$ on $U_j(X)$ and $h(\hat{T}_\theta x) = T_\theta h(x)$ $\forall\, x \in U_j(X)$ and $\theta \in [0,2\pi]$. Let $A = \overline{h(D_{j+1}(X)\backslash Y)}$ then $A \in \mathcal{A}_{k+1}$ and we have that

$$\sup_{z\in A} L(z) = \sup_{z\in \overline{h_1(U_j(X)\backslash Y)}} \left(\max_{\theta\in[0,2\pi]} L(T_\theta x)\right) \le \sup_{z\in B} L(z) \quad .$$

This completes the proof of Lemma 5.9 and then Proposition 5.4.

□

<u>Remark</u>. Proposition 5.4 is a variant of Lemma 1.64 [18] in S^1-setting. Lemma 5.9 is new. The space X is introduced to get the unique expression $y = \hat{T}_\theta x$ for given $y \in D_{j+1}(X)$ in terms of (x,θ) in $U_j(X) \times [0,2\pi)$, which is crucial in the proof of Lemma 5.9.

§6. A lower estimate for the growth rate of $\{a_k\}$

In this section, we shall prove the following estimate on $\{a_k\}$.

<u>Proposition 6.1</u>. There are constants $\lambda > 0$, $k_0 > N+1$ such that

(6.2) $$a_k \geq \lambda k (\log k)^{2/q_2} \quad \forall k \geq k_0 .$$

We shall carry out the proof in several steps.

Step 1. We consider a Hamiltonian system

(6.3) $$\dot{z} = J\nabla F(|z|) \equiv \frac{F'(|z|)}{|z|} J z$$

and its corresponding Lagrangian functional

$$\Phi_F(z) = \frac{1}{2} A(z) - \int_0^{2\pi} F(|z|)dt \quad \forall z \in C^1(S^1, R^{2N}) .$$

By direct computation we have

Lemma 6.4. If the function F satisfies

(F1) $F \in C^1([0,+\infty), R)$.

(F2) Let $g_F(t) = \frac{F'(t)}{t}$, then $g_F(0) = 0$, $\lim_{t\to+\infty} g_F(t) = +\infty$, and $g_F(t)$ is strictly increasing.

(F3) Let $h_F(t) = F'(t)t - 2F(t)$, then $h_F(g_F^{-1}(1)) > 0$ and $h_F(t)$ is strictly increasing for $t \geq g_F^{-1}(1)$.

Then

1°. The solutions of (6.3) are all in E^+ and of the following form

$$z_k(t) = \begin{pmatrix} \cos(kt)I & \sin(kt)I \\ \sin(kt)I & -\cos(kt)I \end{pmatrix} v_k$$

for any $v_k \in \mathbb{R}^{2N}$ with $|v_k| = \gamma_k(F)$, $k \in \mathbb{N}$ and $v_0 = 0$ where $\gamma_k(F) = g_F^{-1}(k)$ satisfies $\gamma_k(F) \to +\infty$ as $k \to +\infty$ and $0 = \gamma_0(F) < \gamma_k(F) < \gamma_{k+1}(F)$ for $k \in \mathbb{N}$, I is the identity matrix on $\mathbb{R}^N$.

2°. Let $d_0(F) = 0$, $d_k(F) = \Phi_F(z_k)$ $\forall k \in \mathbb{N}$, then $d_k(F) = \pi h_F(\gamma_k) > 0$, is strictly increasing in k.

Step 2. We define a function $G : [0,+\infty) \to R$ by

$$G(t) = \alpha \sum_{k=n+1}^{\infty} \frac{\tau^k}{k!} t^{kq} = \alpha e^{\tau t^q} - \sum_{k=0}^{n} \frac{\tau^k}{k!} t^{kq}$$

where $\alpha = 2\alpha_2$, $\tau = \tau_2$, $q = q_2$, α_2, τ_2, q_2 are given by (H3), and n is the smallest positive integer such that $n \geq [\frac{5}{q}] + 1$ and $\tau\alpha q(\frac{\tau q}{4})^{2/q} \sum_{k=n}^{\infty} \frac{1}{k!} (\frac{4}{q})^k < 1$.

Then it is easy to see G satisfies (F1) and (F2). Since

$$(6.5)\quad h_G(t) \equiv G'(t)t - 2G(t) = \alpha q \frac{\tau^{n+1}}{n!} t^{(n+1)q} + \alpha(\tau q t^q - 2) \sum_{k=n+1}^{\infty} \frac{\tau^k}{k!} t^{kq}$$

when $t > \left(\frac{2}{\tau q}\right)^{1/q}$, $h_G(t) > 0$ and is strictly increasing.

Write $\gamma_k = \gamma_k(G)$ for $k \in \mathbf{N}$, we claim that $\gamma_1 > \left(\frac{4}{\tau q}\right)^{1/q}$. For otherwise we have the following contradiction,

$$1 = g_G(\gamma_1) = \alpha q \sum_{k=n+1}^{\infty} \frac{\tau^k}{(k-1)!} \gamma_1^{kq-2} < \tau\alpha q\left(\frac{\tau q}{4}\right)^{2/q} \sum_{k=n}^{\infty} \frac{1}{k!} \left(\frac{4}{q}\right)^k < 1 \quad .$$

Therefore G satisfies (F3), and we also have that

$$(6.6)\qquad G'(t)t - 4G(t) = \alpha q \frac{\tau^{n+1}}{n!} t^{(n+1)q} + \alpha(\tau q t^q - 4) \sum_{k=n+1}^{\infty} \frac{\tau^k}{k!} t^{kq} > 0$$

and is strictly increasing for $t > \left(\frac{4}{\tau q}\right)^{1/q}$. It is also easy to see that there is $t_1 > 0$ such that

$$(6.7)\qquad 0 < G(t) < t^4 \quad \forall\, t \in [0,t_1] \quad .$$

We consider the Hamiltonian system

$$(6.8)\qquad \dot{z} = J\nabla G(|z|) \equiv \frac{G'(|z|)}{|z|} Jz$$

and write $\Phi = \Phi_G$, $d_k = d_k(G)$ $\forall\, k \in \{0\} \cup \mathbf{N}$. Besides properties described in Lemma 6.4 we also have

<u>Lemma 6.9</u>. There are $\lambda_1 > 0$, $k_1 \in \mathbf{N}$ such that

$$(6.10)\qquad d_k > \lambda_1 k (\log k)^{2/q} \quad \forall\, k > k_1 \quad .$$

<u>Proof</u>. From $g_G(\gamma_k) = k$ we have

$$\tau\alpha q\, \gamma_k^{q-2} e^{\tau\gamma_k^q} - \alpha q \sum_{j=1}^{n} \frac{\tau^j}{(j-1)!} \gamma_k^{jq-2} = k$$

so

$$\tau\gamma_k^q > \log k + (2-q)\log \gamma_k - \log(\alpha\tau q) \quad .$$

Thus there is $k_2 \in \mathbf{N}$ such that

$$\gamma_k > \left(\frac{1}{2\tau} \log k\right)^{1/q} \quad \forall\, k > k_2 \quad .$$

Since $G(t) = (\frac{G'(t)}{t} - \tau\alpha q t^{(n+1)q-2} \frac{\tau^n}{n!}) \frac{1}{\tau q} t^{2-q}$, we get

$$d_k = \pi(G'(\gamma_k)\gamma_k - 2G(\gamma_k)) > \pi(k\gamma_k^2 - \frac{2k}{\tau q}\gamma_k^{2-q}) = \pi k\gamma_k^2(1 - \frac{2}{\tau q}\gamma_k^{-q}) .$$

So there is $k_1 > k_2$ such that for any $k > k_1$

$$d_k > \frac{\pi}{2} k\gamma_k^2 > \lambda_1 k (\log k)^{2/q}$$

where $\lambda_1 = \frac{\pi}{2}(\frac{1}{2\tau})^{2/q}$.

For $k \in \mathbb{N}$, $k > N+1$ we define

$$c_k = \inf_{a\in A_k} \sup_{z\in A} \Phi(z) .$$

Since $\Phi \in C(E,\mathbb{R})$, (6.6) and $0 \in \bigcap_{A\in A_k} A$, we get $-\infty < c_k < +\infty$. From $A_{k+1} \subset A_k$ we get

(6.11) $$c_k < c_{k+1} \quad \forall\, k > N+1 .$$

By (H3) and the definition of G, there is a constant $\zeta_1 > 0$ such that

$$H(z) + \frac{1}{2}|z|^2 < G(|z|) + \zeta_1 \qquad \forall\, z \in \mathbb{R}^{2N} .$$

For $z \in E$, by Hölder's inequality we get for $\zeta_2 = \zeta_1 + \frac{1}{2}\|f\|^2_{L^2}$.

$$J(z) > \frac{1}{2}A(z) - \int_0^{2\pi}(H(z) + \frac{1}{2}|z|^2)dt - \frac{1}{2}\|f\|^2_{L^2} > \Phi(z) - \zeta_2 .$$

So we get

(6.12) $$a_k > c_k - \zeta_2 \qquad \forall\, k > N+1 .$$

<u>Step 3.</u> Let $m_0 = [\gamma_1] + 1$ and define for $m > m_0$

$$G_m(t) = \begin{cases} G(t) & , \text{ if } 0 < t < m \\ \frac{G'(m)}{4m^3} t^4 + G(m) - \frac{m}{4} G'(m) & , \text{ if } m < t . \end{cases}$$

Then G_m satisfies (F1) and (F2). For $n < t$

(6.13) $$G_m'(t)t - 2G_m(t) = \frac{G'(m)}{m^3} t^4 + \frac{1}{2}(G'(m)m - 4G(m)) .$$

By (6.6) and the properties of G, G_m satisfies (F3). So Lemma 6.4 holds for G_m. Since

$$G_m'(t)t - 4G_m(t) = \frac{G'(m)}{m^3} t^4 + (G'(m)m - 4G(m)) \quad , \quad \forall\, t > m \quad ,$$

We get that

$$0 < G_m(t) \leq G_{m+1}(t) \leq G(t) \quad \forall\, t \geq 0$$

and

$$0 < 4G_m(t) \leq G_m'(t)t \quad \forall\, t \geq r_1 \equiv (\frac{4}{\tau q})^{1/q} \quad .$$

From the definition of G_m and (6.7), there is a constant $\sigma_m > 0$ depending on m such that

(6.14) $$G_m(t) \leq \sigma_m t^4 \quad \forall\, t \geq 0 \quad .$$

We consider

(6.15) $$\dot{z} = J\nabla G_m(|z|) \equiv \frac{G_m'(|z|)}{|z|} Jz$$

and write $\Phi_m = \Phi_{G_m}$, $d_k(m) = d_k(G_m)$. Then $\Phi_m \in C^1(E,R)$ and satisfies (P.S) condition. Define

$$c_k(m) = \inf_{A\in A_k} \sup_{z\in A} \Phi_m(z) \quad \forall\, k \geq N+1,\ m \geq m_0 \quad .$$

Then we have $-\infty < c_k(m) < +\infty$. To get more accurate estimates on $c_k(m)$, we need

<u>Lemma 6.16</u>. Let $N+1 \leq j \leq m$, $0 < \rho < R_m$, $h \in \Gamma_m$ and

$$Q \equiv \{x \in D_m(X) \mid h(x) \in \partial B_\rho(E) \cap V_{j-1}^{\perp}(E)\} \quad .$$

Then Q is compact and $\gamma(Q) \geq m-j+1$.

<u>Proof</u>. This Lemma is a variant of Proposition 1.19 [19]. Note that firstly $id(x_i^-) = P^- id(x_i)$ by 5° of Lemma 3.3, $\{P^- id(x_i)\}$ being convergent implies that $\{x_i^-\}$ has a convergent subsequence. This yields that Q is compact. Secondly, by the definition of $h \in \Gamma_m$, if we let $E_k = E_{N+1,k}^+ \oplus E_{N+1,k}^- \oplus E^0$, $X_k = X_{N+1,k}^+ \oplus X_{N+1,k}^- \oplus X^0$ and $P_k : E \to E_k$ be the orthogonal projector, then $P_k h(x) = z$ for $z \sim x \in X^0 \cap D_m(X)$ and $P_k h(\partial B_m(X) \cap V_m(X) \cap X_k) =$

$P_k id(\partial B_m(X) \cap V_m(X) \cap X_k) = \partial B_m(E) \cap V_m(E) \cap E_k$. This allows us to apply Borsuk-Ulam theorem [8]. Therefore we can go through the proof of Proposition 1.19 [19]. We omit the details here.

□

Corollary 6.17. Let $N+1 \leq j \leq m$, $0 < \rho < R_m$, $h \in \Gamma_m$. For any $Y \in X$ with $\gamma(Y) \leq m-j$

$$\overline{h(D_m(X)\backslash Y)} \cap \partial B_\rho(E) \cap V_{j-1}^{\perp}(E) \neq \emptyset \quad .$$

Lemma 6.18. $c_k(m) > 0 \quad \forall\, k \geq N+1,\ m \geq m_0$.

Proof. Fix $m \geq m_0$, $k \geq N+1$, by Corollary 6.17 for any $A \in \mathcal{A}_k$ and $0 < \rho < R_k$ there is a $z \in A \cap \partial B_\rho \cap E^+$. Let C denote the embedding constant from E into L^4, then by (6.14)

$$\Phi_m(z) \geq \frac{1}{2} A(z) - \sigma_m \int_0^{2\pi} |z|^4 dt \geq \frac{1}{2}\rho^2 - \sigma_m C\rho^4 = \frac{1}{2}\rho^2(1 - \sigma_m C\rho^2) \quad .$$

Choose $\rho_m = \min\{1, (2\sigma_m C)^{-1/2}\}$, we get $c_k(m) \geq \frac{1}{2}\rho_m^2 > 0$.

□

Lemma 6.19. For any $k \geq N+1$, $m \geq m_0$

1°. $c_k(m)$ is a critical value of Φ_m.

2°. Any critical point of Φ_m corresponding to $c_k(m)$ lies in $E\backslash E^0$.

3°. If $c_{k+1}(m) = \cdots = c_{k+j}(m) \equiv c$ and $K \equiv (\Phi_m')^{-1}(0) \cap \Phi_m^{-1}(c)$, then $\gamma(K) \geq j$.

Proof. 1° and 3° follow from the standard argument, we refer to [19]. 2° follows from 1°, Lemma 6.4 and Lemma 6.18. We omit details here.

□

Lemma 6.20. For $k \in \mathbf{N}$, $k \geq N+1$, $m \geq m_0$, $c_k \leq c_k(m+1) \leq c_k(m)$ and

$$\lim_{m\to\infty} c_k(m) = c_k$$

Proof. The Lemma follows from the proofs of Lemmas 4.1 and 4.2.

□

Step 4. Proof of Proposition 6.1.

Fix $k \geq N+1$, for any $m \geq m_0$, by 1° of Lemma 6.19 and Lemma 6.18, $c_k(m) = d_j(m)$ for some $j > 0$. So

$$c_k(m) = \Phi_m(z_j) = \pi(G_m'(\gamma_j)\gamma_j - 2G_m(\gamma_j)) \geq \min\{\pi \frac{G'(m)}{m^3} \gamma_j^4, \pi\alpha q \frac{\tau^{n+1}}{n!} \gamma_j^{(n+1)q}\} ,$$

here we used (6.5) and (6.13). Since by definition of G, $\frac{G'(t)}{t^3}$ is strictly increasing for $t > 0$, by Lemma 6.20 we get

$$c_k(m_0) \geq c_k(m) \geq \min\{\pi \frac{G'(m_0)}{m_0^3} \gamma_j^4, \pi\alpha q \frac{\tau^{n+1}}{n!} \gamma_j^{(n+1)q}\} .$$

So there exists $M_1 > 0$ independent of m such that if z is a critical point of Φ_m corresponding to $c_k(m)$ with $m \geq m_0$, then $\|z\|_C \leq M_1$. Thus $G_m(|z|) = G(|z|)$ for $m \geq m_1(k) \equiv \max\{m_0, [M_1] + 1\}$, and then there exists $j(m) \in \mathbf{N}$ depending on m such that

(6.21) $$c_k(m) = d_{j(m)} \quad \forall\, m \geq m_1(k) .$$

By Lemma 6.20, $0 < d_k < d_{k+1}$, and (6.10), we get $c_k = d_j$ for some $j \in \mathbf{N}$. Therefore $\{c_k\}$ is a subset of $\{d_k\}$.

We claim that $c_{k+N} > c_k \;\; \forall\, k \geq N+1$. If not, by (6.11) we get $c \equiv c_k = \cdots = c_{k+N}$. By (6.21) there exists $m \geq m_0$ depending on k such that $c = c_k(m) = \cdots = c_{k+N}(m)$. Let $K = (\Phi_m')^{-1}(0) \cap \Phi_m^{-1}(c)$. By 3° of Lemma 6.19, $\gamma(K) \geq N+1$. But 1° of Lemma 6.4 and 4° of Lemma 3.2 shows that $\gamma(K) = N$. This contradiction proves the claim.

Assume $c_{N+1} = d_\ell$ for some $\ell \in \mathbf{N}$, then by the above discussion and (6.10), for $k \geq \max\{k_1, 6N\}$

$$c_k > c_{N+1+[\frac{k-N-2}{N}]N} > d_{\ell+[\frac{k-N-2}{N}]} > \lambda_1(\ell + [\frac{k-N-2}{N}])\log^{2/q}(\ell + [\frac{k-N-2}{N}])$$

$$> \lambda_1(\ell + \frac{k}{N} - 3)\log^{2/q}(\ell + \frac{k}{N} - 3) > \lambda_1 \cdot \frac{k}{2N} \cdot \log^{2/q}(\frac{k}{2N})$$

$$> \lambda k (\log k)^{2/q}$$

for some $\lambda > 0$. Combining with (6.12) we get (6.2).

The proof of Proposition 6.1 is complete.

□

§7. The existence of critical values of J_n

Fix $n, k \in \mathbb{N}$, $k > N+1$, we have

Proposition 7.1. Suppose $b_k(n) > a_k(n) > \beta_8$. Let $\delta_k(n) \in (0, b_k(n) - a_k(n))$ and

$$\mathcal{B}_k(n,\delta_k(n)) = \{\overline{h(D_{j+1}(X)\backslash Y)} \in \mathcal{B}_k \mid J_n(h(x)) < a_k(n) + \delta_k(n) \text{ for } x \in D_j(X)\backslash Y\} .$$

Let

$$b_k(n,\delta_k(n)) = \inf_{B\in\mathcal{B}_k(n,\delta_k(n))} \sup_{z\in B} J_n(z) .$$

Then $b_k(n,\delta_k(n))$ is a critical value of J_n.

Remark. $b_k(n,\delta_k(n)) > b_k(n)$. By Lemma 3.6, $\mathcal{B}_k(n,\delta_k(n)) \neq \emptyset$, and $b_k(n,\delta_k(n)) < +\infty$.

For the proof of Proposition 7.1, we need the following "Deformation Theorem", which was proved in [19].

Lemma 7.2. Let J_n be as above, then if $b > \beta_8$, $\bar{\varepsilon} > 0$ and b is not a critical value of J_n, there exist $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

1°. $\eta(t,z) = z$ if $z \notin J_n^{-1}(b-\bar{\varepsilon}, b+\bar{\varepsilon})$

2°. $\eta(0,z) = z \quad \forall\, z \in E$

3°. $\eta(1, [J_n]^{b+\varepsilon}) \subset [J_n]^{b-\varepsilon}$, where $[J_n]^a = \{z \in E \mid J_n(z) < a\}$.

4°. $P^-\eta(1,z) = \alpha_1(z)z^- + \beta_1(z) \quad \forall\, z \in E$, where $\alpha_1 \in C(E,[1,e^2])$,

$\beta_1 \in C(E,E^-)$ and β is compact.

5°. $\eta(t,\cdot)$ is a bounded map from E to E for $t \in [0,1]$.

Proof of Proposition 7.1. Let $\bar{\varepsilon} = \frac{1}{2}(b_k(n) - a_k(n)) > 0$. If $b_k(n,\delta_k(n))$ is not a critical value of J_n, then there exist ε and η as in Lemma 7.2. Choose $B \in \mathcal{B}_k(n,\delta_k(n))$ such that

$$\sup_{z\in B} J(z) < b_k(n,\delta_k(n)) + \varepsilon \quad .$$

Then there exist $j > k$, $h_0 \in \Lambda_j$, $Y \in \mathcal{X}$ with $\gamma(Y) < j-k$ such that $B = \overline{h_0(D_{j+1}(X)\backslash Y)}$. Define

$$h(x) = \eta(1,h_0(x)) \quad \forall\, x \in \overline{D_{j+1}(X)\backslash Y} \equiv Q_1$$

$$h(x) = h_0(x) \quad \forall\, x \in B_{j+1}(X) \cap V_j(X) \cap Y \equiv Q_2$$

$$h(x) = id(x) \quad \forall\, x \in \partial B_{j+1}(X) \cap V_{j+1}(X) \equiv Q_3.$$

Denote $Q = Q_1 \cup Q_2 \cup Q_3$.

For $x \in \overline{D_j(X)\backslash Y}$, $J_n(h_0(x)) < a_k(n) + \delta_k(n) < b_k(n) - 2\bar{\varepsilon} < b_k(n,\delta_k(n)) - \bar{\varepsilon}$.

Thus

$$h(x) \equiv \eta(1,h_0(x)) = h_0(x) \quad \forall\, x \in \overline{D_j(X)\backslash Y} \quad . \tag{7.3}$$

For $x \in Q_3 \cup ((B_{j+1}(X)\backslash B_j(X)) \cap V_j(X))$, $J_n(h_0(X)) < 0 < b_k(n,\delta_k(n)) - \bar{\varepsilon}$, thus $\eta(1,h_0(x)) = h_0(x) = id(x)$. So $h \in C(Q,E)$.

For $x \in Q_4 \equiv (D_{j+1}(X) \cap V_j(X)) \cup Q_3$, $P^-(h(x) = P^-h_0(x) = \alpha_0(x)P^-id(x) + \beta_0(x)$ where α_0, β_0 are defined for h_0 in 3° of Definition 3.5. For $x \in Q\backslash Q_4$,

$$P^-h(x) = \alpha_1(h_0(x))\alpha_0(x)P^-id(x) + \alpha_1(h_0(x))\beta_0(x) + \beta_1(h_0(x)) .$$

Define

$$\alpha(x) = \begin{cases} \alpha_0(x) & \text{if } x \in Q_4 , \\ \alpha_1(h_0(x))\alpha_0(x) & \text{if } x \in Q\backslash Q_4 . \end{cases}$$

$$\beta(x) = \begin{cases} \beta_0(x) & , \text{ if } x \in Q_4 , \\ \alpha_1(h_0(x))\beta_0(x) + \beta_1(h_0(x)) , & \text{if } x \in Q\backslash Q_4 . \end{cases}$$

Then $\alpha \in C(Q,[1,e^{2\tilde{\alpha}_0}])$, $\beta \in C(Q,E^-)$ is compact and

$$P^-h(x) = \alpha(x)P^-id(x) + \beta(x) \quad \forall\, x \in Q .$$

Let $W = (D_{j+1}(X) \cap Y)\backslash V_j(X)$, then $\partial W \subset Q$ where "∂" is taken within $V_{j+1}(X)$. Since α, β, P^+h, and P^0h are continuously defined on ∂W, we may use the Dugundji extension theorem ([7]) to extend them to $\overline{W}$, then define $P^-h(x) = \alpha(x)P^-id(x) + \beta(x)$, and $h(x) = P^+h(x) + P^-h(x) + P^0h(x)$. We have $h \in \Lambda_j$. Thus $D \equiv \overline{h(D_{j+1}(X)\backslash Y)} \in \mathcal{B}_k$. By (7.3)

$$J_n(h(x)) = J_n(h_0(x)) < a_k(n) + \delta_k(n) \quad \forall\, x \in \overline{D_j(X)\backslash Y} .$$

Thus $D \in \mathcal{B}_k(n,\delta_k(n))$. Now 3° of Lemma 7.2 yields

$$\sup_{z\in D} J_n(z) < b_k(n,\delta_k(n)) - \varepsilon .$$

This contradicts to the definition of $b_k(n,\delta_k(n))$. Therefore the proof is complete.

□

§8. The proofs of the main theorems

We prove Theorem 1.2 by contradiction. Assume that the functional I is bounded from above by $M_1 > 0$ on S, the solution set of (1.1).

Since $q_2 < 2q_1$, Propositions 5.4 and 6.1 show that there exists $k \in \mathbb{N}$ such that

$$b_k > a_k \geq \max\{\beta_8, M_1\} .$$

Let $\varepsilon = \frac{1}{5}(b_k - a_k)$. By Lemmas 4.1 and 4.2, there exists $n_1 > 0$ such that

$$b_k(n) - a_k(n) \geq 4\varepsilon \quad \text{and} \quad a_k(n) \leq a_k + \varepsilon \quad \forall\, n \geq n_1 .$$

Let $\delta_k(n_1) = \varepsilon$, and $\delta_k(n) = 2\varepsilon$ for $n > n_1$. Then by Proposition 7.1, $b_k(n, \delta_k(n))$ is a critical value of J_n for $n \geq n_1$.

If $\overline{h(D_{j+1}(X)\backslash Y)} \in \mathcal{B}_k(n_1, \varepsilon)$ then for any $x \in D_j(X)\backslash Y$

$$J_n(h(x)) \leq J_{n_1}(h(x)) \leq a_k(n_1) + \varepsilon \leq a_k + 2\varepsilon \leq a_k(n) + 2\varepsilon \quad \forall\, n > n_1 .$$

So $\mathcal{B}_k(n_1, \varepsilon) \subset \mathcal{B}_k(n, 2\varepsilon) \quad \forall\, n > n_1$. Therefore for $n > n_1$,

$$b_k(n, 2\varepsilon) \leq \inf_{B \in \mathcal{B}_k(n_1, \varepsilon)} \sup_{z \in B} \left(\frac{1}{2} A(z) - \int_0^{2\pi} P_0(z)dt + \int_0^{2\pi} |f \cdot z| dt\right) \equiv b < +\infty$$

where $P_0(z) = \alpha_0 |z|^{\mu\sigma} - \beta_0$ and we used (2.6).

Let z_n be a critical point of J_n corresponding to $b_k(n, 2\varepsilon)$ for $n > n_1$. Using (2.6), $f \in W^{1,2}(S^1, \mathbb{R}^{2N})$ and the proof of Lemma 5.3 [2], we get

$$\|z_n\|_{L^\infty} \leq M_2 \quad \forall\, n > n_1 \tag{8.1}$$

where the constant $M_2 > 0$ depending on b, but independent of n. Now we choose $n_2 > n_1$ such that $K_{n_2} > M_2$, where $\{K_n\}$ is defined in Proposition 2.5. From (8.1) and Lemma 2.19 we get that $H'_{n_2}(z_{n_2}) = H'(z_{n_2})$ on $[0, 2\pi]$ and z_{n_2} is a solution of (1.1) i.e. $z_{n_2} \in S$. But

$$I(z_{n_2}) = I_{n_2}(z_{n_2}) = J_{n_2}(z_{n_2}) = b_k(n_2, 2\varepsilon) \geq b_k(n_2) > a_k(n_2) \geq a_k > M_1 .$$

This contradicts to the definition of M_1, and completes the proof of Theorem 1.2.

The proof of Theorem 1.3 is similar. Instead of (5.5) and (6.2) we shall have "$a_k \le Mk^{(p_1+1)/p_1}$, $\forall\, k \ge k_1$" and "$a_k \ge \alpha k^{(p_2+1)/(p_2-1)}$", by (H4) they yield "$b_k > a_k$ for infinitely many k". The proof is rather simpler than that of Theorem 1.2. For example, the corresponding

$$\Phi(z) = \frac{1}{2} A(z) - (\alpha_2 + \frac{1}{2}) \int_0^{2\pi} |z|^{p_2+1} dt$$

is C^2 and satisfies (P.S.) condition. So the lower estimate for a_k is quite straightforward. For the details we refer to [13].

In [15], Pisani and Tucci gave a result for (1.1),

<u>Theorem 8.2</u> (Theorem 1.1 [15]). Let H satisfy (H1) and the following conditions

(H5)
$$\lim_{|z|\to+\infty} \frac{H'(z)\cdot z}{|z|^2} = +\infty \quad .$$

(H6) There are constants $p > 1$, $\alpha_1, \beta_1 > 0$ such that

$$\frac{1}{2} H'(z)\cdot z - H(z) \ge \alpha_1 |z|^{p+1} - \beta_1 \quad \forall\, z \in \mathbb{R}^{2N} .$$

(H7) There are constants $q \in [p,p+1)$ and $\alpha_2, \beta_2 > 0$ such that

$$|H'(z)| \le \alpha_2 |z|^q + \beta_2 \quad \forall\, z \in \mathbb{R}^{2N} \quad .$$

Then the conclusion of Theorem 1.3 holds for given $T > 0$ and T-periodic function $f \in L^2_{\ell oc}(\mathbb{R},\mathbb{R}^{2N})$.

Unfortunately, because of a difficulty caused by the S^1-action on $W^{1/2,2}(S^1,\mathbb{R}^{2N})$, their proof (Lemma 1.14 [15]) is not complete. Using the minimax idea introduced in §3, this difficulty can be overcome. The condition (H7) allows us to carry out the proof without doing any truncation on H, so the function f can be allowed only in L^2 and the proof becomes rather

simpler. We omit the details here.

We also refer readers to a related density result proved earlier.

Theorem 8.3 (Theorem 1.5 [11]). Let H satisfy (H1) and (H5), then for any $T > 0$, there exists a dense set D in the space of T-periodic functions in $L^2([0,T],\mathbf{R}^{2N})$ such that for every $f \in D$, (1.1) is solvable.

This theorem poses a natrual question whether the condition (H3) or (H4) is necessary in Theorems 1.2 or 1.3.

§9. Results for general forced systems

In this section we consider the general Hamiltonian system

$$(9.1) \qquad \dot{z} = J\hat{H}_z(t,z) \quad .$$

Firstly we consider (9.1) with bounded perturbations. That is

Theorem 9.2. Let $\hat{H}$ satisfy the following conditions,

(G1) $\hat{H} \in C^1(\mathbf{R} \times \mathbf{R}^{2N},\mathbf{R})$ and $\hat{H}(t,z)$ is T-periodic in t.

(G2) There exists $H : \mathbf{R}^{2N} \to \mathbf{R}$ satisfying (H1), (H2) and constants $0 < q < 2$ $\alpha, \tau > 0, \beta > 0$ such that

1°. $H(z) \leq \alpha e^{\tau|z|^q} + \beta \quad \forall\, z \in \mathbf{R}^{2N}$.

2°. $|\hat{H}(t,z) - H(z)| \leq \alpha \quad \forall\, (t,z) \in \mathbf{R} \times \mathbf{R}^{2N}$.

3°. $|\hat{H}_z(t,z) - H_z(z)| \leq \alpha\big(|z|^{p-1} + 1\big) \quad \forall\, (t,z) \in \mathbf{R} \times \mathbf{R}^{2N}$, where $1 \leq p < \mu$ and $\mu > 2$ is defined in (H2).

then the system (9.1) possesses infinitely many distinct T-periodic solutions.

Remark. Theorem 9.2 weakened conditions of Bahri and Berestycki's corresponding result, Theorem 10.1 [2], who required $\hat{H}$ satisfying the following conditions

1°. $\hat{H} \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ and is T-periodic in t.

2°. There exists $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ satisfying (H2) and constants $q > 1$, $\alpha > 0$ such that

$$H(z) \le \alpha(|z|^{q+1} + 1) \quad \forall\, z \in \mathbb{R}^{2N}$$

and

$$\|\hat{H} - H\|_{C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})} < \infty \ .$$

In order to prove Theorem 9.2, we let $G(t,z) = \hat{H}(t,z) - H(z)$, and consider functionals

$$J(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H(z)dt - \psi(z) \int_0^{2\pi} G(t,z)dt$$

and

$$J_n(z) = \frac{1}{2} A(z) - \int_0^{2\pi} H_n(z)dt - \psi_n(z) \int_0^{2\pi} G(t,z)dt$$

where H_n, ψ_n are defined in §2, and we can go through the proofs in §§2 - 7 with the following estimates for $\{a_k\}$ from above.

<u>Lemma 9.3</u>. If $b_k = a_k \ \forall\, k \ge k_1$, then there is $M = M(k_1) > 0$ such that

(9.4) $$a_k \le Mk \quad \forall\, k \ge k_1 \ .$$

<u>Proof</u>. Using 2° of (G2), instead of (2.27), we get that $|J(z) - J(T_\theta z)| \le 4\pi\alpha \ \forall\, z \in E$. So as in the proof of Proposition 5.4, we get that

$$a_{k+1} \le \inf_{A\in\mathcal{A}_{k+1}} \sup_{z\in A}\Big(\max_{\theta\in[0,2\pi]} J(T_\theta z)\Big) \le \inf_{b\in\mathcal{B}_k} \sup_{z\in B}\Big(\max_{\theta\in[0,2\pi]} J(T_\theta z)\Big) \text{ (by Lemma 5.9)}$$

$$\le \inf_{B\in\mathcal{B}_k} \sup_{z\in B} J(z) + 4\pi\alpha = b_k + 4\pi\alpha = a_k + 4\pi\alpha \quad \forall\, k \ge k_1 \ .$$

Let $\delta_k = \frac{a_k}{k}$. If $\delta_{k+1} > \delta_k$, then from the above inequality

$$(k+1)\delta_{k+1} \le k\delta_k + 4\pi\alpha \le k\delta_{k+1} + 4\pi\alpha$$

so

$$\delta_{k+1} < 4\pi\alpha \ .$$

This shows that

$$\delta_{k+1} < \max\{\delta_k, 4\pi\alpha\} \quad \forall\ k > k_1 \ .$$

Thus

$$\delta_k < \max\{\delta_{k_1}, 4\pi\alpha\} \quad \forall\ k > k_1 \ .$$

Let $M = \max\{\frac{a_{k_1}}{k_1}, 4\pi\alpha\}$, we get (9.4), and completes the proof of Lemma 9.3.

□

Now the arguments in §8 yield Theorem 9.2.

Secondly, it is not difficult to get direct generalizations of Theorems 1.2 and 1.3 for (9.1).

Theorem 9.5. Let $\hat{H}$ satisfy conditions (G1) and

(G3) There exists $H : \mathbf{R}^{2N} \to \mathbf{R}$ satisfying (H1), (H2) and either

1°. H satisfying (H3) and there are $\alpha > 0$ and $1 < p < \min\{\frac{2q_1}{q_2}, \mu\}$

such that

$$|\hat{H}(t,z) - H(z)| < \alpha(|z|^p+1), \ |\hat{H}_z(t,z) - H_z(z)| < \alpha(|z|^{p-1}+ 1), \ \forall(T,z) \in \mathbf{R} \times \mathbf{R}^{2N}.$$

or

2°. H satisfying (H4) and there are $\alpha > 0$ and $1 < q < \frac{2(p_1+1)}{p_2+1}$ such that

$$|\hat{H}_z(t,z) - H_z(z)| < \alpha(|z|^{q-1} + 1) \quad \forall(t,z) \in \mathbf{R} \times \mathbf{R}^{2N} \ .$$

Then the system (9.1) possesses infinitely many distinct T-periodic solutions.

We omit the details here.

Appendix. Monotone truncations of H in $C^1(\mathbf{R}^{2N},\mathbf{R})$

In this appendix, we give a proof of Proposition 2.5.

Recall that $\sigma \in (0,1)$, $\mu\sigma > 2$, and $r_0 > 1$ (§2). Choose $\lambda \in (\sigma,1)$ such that $\mu(\lambda-\sigma) < 1$. Define $K_1 = K_0 + 2$, $\tau_0 = 1$. For $n \in \mathbf{N}$, define inductively that

$$\tau_n = \max\{\tau_{n-1} + 2,\ \alpha_0 + \frac{1}{K_n^{\mu\sigma}} \max_{K_n \le |z| \le K_n+1} H(z)\} \quad , \tag{A.1}$$

$$K_{n+1} = \max\{K_n + 2,\ (\frac{\tau_n}{\alpha_0})^{\frac{1}{\mu(1-\lambda)}}\} \quad . \tag{A.2}$$

Here $\alpha_0 = \min_{|z|=r_0} H(z) > 0$. For $K \in \mathbf{R}$, take $\chi(\cdot,K) \in C^\infty(\mathbf{R},\mathbf{R})$ such that $\chi(s,K) = 1$ for $s \le K$, $\chi(s,K) = 0$ for $s \ge K+1$ and $\chi'(s,K) < 0$ for $s \in (K,K+1)$. Then for $n \in \mathbf{N}$ set

$$M_n(z) = \chi(|z|,K_n)H(z) + (1 - \chi(|z|,K_n))\tau_n|z|^{\mu\lambda} \quad \forall\, z \in \mathbf{R}^{2N} \quad .$$

This kind of truncation functions was used by Rabinowitz in [17]. Since the M_n's do not satisfy 4° of Proposition 2.5, we need to modify them. Direct computations (cf. [17]) show that

Lemma A.3. For $n \in \mathbf{N}$, $M_n \in C^1(\mathbf{R}^{2N},\mathbf{R})$ has the following properties,

$$M_n(z) = H(z) \quad \text{if} \quad |z| \le K_n \quad , \tag{A.4}$$

$$M_n(z) = \tau_n|z|^{\mu\lambda},\ M_n'(z)\cdot z = \tau_n\mu\lambda|z|^{\mu\lambda}, \quad \text{if} \quad |z| \ge K_n + 1 \quad , \tag{A.5}$$

$$0 < \mu\lambda M_n(z) \le M_n'(z)\cdot z \quad \text{for} \quad |z| \ge r_0 \quad . \tag{A.6}$$

Integrating (A.6) we get that

$$\alpha_0|z|^{\mu\lambda} \le M_n(z) \quad \text{for} \quad |z| \ge r_0 \quad . \tag{A.7}$$

Lemma A.8. For $\rho \ge K_{n+1}$ we have that

$$\max_{z\in S^{2N-1}} M_n'(\rho z)\cdot z \le \min_{z\in S^{2N-1}} H'(\rho z)\cdot z \tag{A.9}$$

and

$$\text{(A.10)} \qquad \max_{z\in S^{2N-1}} M_n'(\rho z)\cdot z < \min_{z\in S^{2N-1}} M_{n+1}'(\rho z)\cdot z \ .$$

Proof. For any $\zeta,\ z \in S^{2N-1}$, by (H2) and (2.4)

$$\begin{aligned} H'(\rho\zeta)\cdot\zeta &> \frac{\mu}{\rho} H(\rho\zeta) > \mu\alpha_0\rho^{\mu-1} \\ &> \mu\lambda\rho^{\mu-1}\alpha_0\rho^{\mu(1-\lambda)} \\ &> \mu\lambda\rho^{\mu\lambda-1}\ \tau_n \quad \text{(by (A.2))} \\ &= M_n'(\rho z)\cdot z \ . \quad \text{(by (A.5))} \end{aligned}$$

This proves (A.9).

For any $\zeta,\ z \in S^{2N-1}$,

$$\begin{aligned} M_{n+1}'(\rho\zeta)\cdot\zeta &= \chi(\rho,K_{n+1})H'(\rho\zeta)\cdot\zeta + (1 - \chi(\rho,K_{n+1}))\tau_{n+1}\mu\lambda\rho^{\mu\lambda-1} \\ &\qquad + \chi'(\rho,K_{n+1})(H(\rho\zeta) - \tau_{n+1}\rho^{\mu\lambda}) \\ &> \min\{H'(\rho\zeta)\cdot\zeta,\ \tau_{n+1}\mu\lambda\rho^{\mu\lambda-1}\} \\ &\qquad \text{(by (A.1) and the definition of } \chi) \\ &> M_n'(\rho z)\cdot z \qquad \text{(by (A.9) and (A.1))} \ . \end{aligned}$$

This proves (A.10).

□

We now introduce the spherical coordinates (r,θ) on $\mathbb{R}^{2N}$. For $z = (z_1,\ldots,z_{2N}) \in \mathbb{R}^{2N}$, write $z = r\bar{z}(\theta)$, $r = |z|$, $\bar{z}(\theta) = \frac{z}{|z|}$, $\theta = (\theta_1,\ldots,\theta_{2N-1})$,

$$\text{(A.11)} \qquad \begin{cases} z_1 = r\cos\theta_1 \\ z_2 = r\sin\theta_1\cos\theta_2 \\ \quad\cdots\cdots \\ z_{2N-1} = r\sin\theta_1\ldots\sin\theta_{2N-2}\cos\theta_{2N-1} \\ z_{2N} = r\sin\theta_1\ldots\sin\theta_{2N-2}\sin\theta_{2N-1} \ , \end{cases}$$

where $r > 0$, $\theta_1 \in [0,\pi]$, $\theta_i \in \mathbb{R}$ for $i = 2,\ldots,2N-1$. We also write $d\theta =$

$d\theta_1 \cdots d\theta_{2N-1}$, and $d\hat{\theta}_i = d\theta_1 \cdots d\theta_{i-1} d\theta_{i+1} \cdots d\theta_{2N-1}$ for $i = 1,\dots,2N-1$.

Let $\Omega = [0,\pi] \times R^{2N-2}$. For $\theta \in \Omega$, $\rho > 0$ define

$$U(\theta,\rho) = \left(\left[\theta_1 - \pi\sqrt{\rho}, \theta_1 + \pi\sqrt{\rho}\right] \cap [0,\pi]\right) \times \prod_{i=2}^{2N-1} \left[\theta_i - \sqrt{\rho}, \theta_i + \sqrt{\rho}\right] .$$

Since H and M_n are uniformly continuous on $\{K_n \le |z| \le K_{n+1}\}$, there is a constant $\delta_n \in (0,1]$ such that

$$\text{(A.12)} \qquad |H(z) - H(\tilde{z})| + |M_n(z) - M_n(\tilde{z})| \le (\lambda - \sigma)\alpha_0 K_n^{\mu\lambda}$$

for any $z, \tilde{z} \in \{K_n \le |z| \le K_{n+1}\}$ and $|z - \tilde{z}| \le \delta_n$. There is a constant $\varepsilon_n \in [0,1]$ such that

$$\text{(A.13)} \qquad |r\overline{z}(\theta) - r\overline{z}(\xi)| \le \delta_n$$

for any $r \in [K_n, K_{n+1}]$, $\theta \in \Omega$ and $\xi \in U(\theta,\varepsilon_n)$. Define

$$\text{(A.14)} \qquad \nu_n(t) = \min\left\{\sqrt{t}, \sqrt{\varepsilon_n}\right\} \quad \text{for } t \ge 0 .$$

For $n \in \mathbb{N}$; $i = 2,\dots,2N-1$; $k = 1,2$; $\theta \in \Omega$; $\rho \ge K_n$ define

$$\omega_{n,1}(\theta_1,\rho) = [\theta_1 - \tfrac{1}{2}\theta_1\nu_n(\rho - K_n), \theta_1 + \tfrac{1}{2}(\pi - \theta_1)\nu_n(\rho - K_n)] ,$$

$$\omega_{n,i}(\theta_i,\rho) = [\theta_i - \tfrac{1}{2}\nu_n(\rho - K_n), \theta_i + \tfrac{1}{2}\nu_n(\rho - K_n)] ,$$

$$\omega_{n,1}^{(k)}(\theta_1,\rho) = [\theta_1 - \tfrac{k}{2}\pi\nu_n(\rho - K_n), \theta_1 + \tfrac{k}{2}\pi\nu_n(\rho - K_n)] \cap [0,\pi] ,$$

$$\omega_{n,i}^{(k)}(\theta_i,\rho) = [\theta_i - \tfrac{k}{2}\nu_n(\rho - K_n), \theta_i + \tfrac{k}{2}\nu_n(\rho - K_n)] ,$$

and for $j = 1,\dots,2N-1$, define

$$\Omega_n(\theta,\rho) = \prod_{j=1}^{2N-1} \omega_{n,j}(\theta_j,\rho) ,$$

$$\Omega_{n,j}(\theta,\rho) = \prod_{\substack{1 \le \ell \le 2N-1 \\ \ell \ne j}} \omega_{n,\ell}(\theta_\ell,\rho) ,$$

$$\Omega_n^{(k)}(\theta,\rho) = \prod_{j=1}^{2N-1} \omega_{n,j}^{(k)}(\theta_j,\rho) ,$$

$$\Omega_{n,j}^{(k)}(\theta,\rho) = \prod_{\substack{1\leq \ell \leq 2N-1 \\ \ell \neq j}} \omega_{n,\ell}(\theta_\ell,\rho) \quad .$$

Then direct computations show that

$$\text{(A.15)} \quad \begin{cases} \omega_{n,1}(\theta_1,\rho) \subseteq [0,\pi] \quad , \\ \Omega_n(\theta,\rho) \subseteq \Omega_n^{(1)}(\theta,\rho) \subseteq \Omega_n^{(2)}(\theta,\rho) \subseteq U(\theta,\rho - K_n) \quad , \\ |\omega_{n,1}(\theta_1,\rho)| = \frac{\pi}{2} \nu_n(\rho - K_n) \quad , \\ |\omega_{n,i}(\theta,\rho)| = \nu_n(\rho - K_n), \ i = 2,\ldots,2N-1 \quad , \\ |\Omega_n(\theta,\rho)| = \frac{\pi}{2} (\nu_n(\rho - K_n))^{2N-1} \quad . \end{cases}$$

To simplify the notations we write $V_n(\rho) = |\Omega_n(\theta,\rho)|$. These sets satisfy:

<u>Lemma A.16</u>. For $n \in \mathbf{N}$, $\theta \in \Omega$, $\rho > K_n$,

1°. $\beta \in \Omega_n(\theta,\rho)$ implies $\theta \in \Omega_n^{(1)}(\beta,\rho)$.

2°. $\beta \in \Omega_n(\theta,\rho)$ and $\gamma \in \Omega_n^{(1)}(\beta,\vartheta)$ imply $\gamma \in \Omega_n^{(2)}(\theta,\rho)$.

<u>Proof</u>. 1°. If $\beta_1 \in \omega_{n,1}(\theta_1,\rho)$, we have

$$\theta_1 - \frac{1}{2}\theta_1\nu_n(\rho - K_n) < \beta_1 < \theta_1 + \frac{1}{2}(\pi - \theta_1)\nu_n(\rho - K_n) \quad ,$$

then

$$\beta_1 - \frac{1}{2}\nu_n(\rho - K_n) < \theta_1(1 - \frac{1}{2}\nu_n(\rho - K_n)) < \theta_1$$

$$< \theta_1 + \frac{1}{2}(\pi - \theta_1)\nu_n(\rho - K_n) < \beta_1 + \frac{\pi}{2}\nu_n(\rho - K_n) \ .$$

Since $\theta_1 \in [0,\pi]$, $\theta_1 \in \omega_{n,1}^{(1)}(\beta_1,\rho)$. Similarly $\beta_i \in \omega_{n,i}(\theta_i,\rho)$ implies $\theta_i \in \omega_{n,i}^{(1)}(\beta_i,\rho)$ for $i = 2,\ldots,2N-1$. Therefore 1° holds.

2°. If $\beta \in \Omega_n(\theta,\rho)$, $\gamma \in \Omega_n^{(1)}(\beta,\rho)$ then

$$\theta_1 - \frac{1}{2}\theta_1\nu_n(\rho - K_n) < \beta_1 < \theta_1 + \frac{1}{2}(\pi - \theta_1)\nu_n(\rho - K_n)$$

and

$$\beta_1 - \frac{\pi}{2}\nu_n(\rho - K_n) < \gamma_1 < \beta_1 + \frac{\pi}{2}\nu_n(\rho - K_n) \quad .$$

So

$$\theta_1 - \pi\nu_n(\rho - K_n) < \theta_1 - \frac{1}{2}(\pi + \theta_1)\nu_n(\rho - K_n) < \beta_1 - \frac{\pi}{2}\nu_n(\rho - K_n) < \gamma_1 \quad ,$$

and

$$\gamma_1 < \beta_1 + \frac{\pi}{2}\nu_n(\rho - K_n) < \theta_1 + \pi\nu_n(\rho - K_n) - \frac{1}{2}\theta_1\nu_n(\rho - K_n) < \theta_1 + \pi\nu_n(\rho - K_n) \quad .$$

Since $\gamma_1 \in [0,\pi]$, we get that $\gamma_1 \in \omega^{(2)}_{n,1}(\theta_1,\rho)$. Similarly $\gamma_i \in \omega^{(2)}_{n,i}(\theta_i,\rho)$ for $i = 2,\ldots,2N-1$. Thus 2° holds and the proof is complete.

□

For $n \in \mathbf{N}$, $\rho > K_n$, $\bar{z} \in S^{2N-1}$ define

$$F_n(\rho,\bar{z}) = \min\{M_n'(\rho\bar{z})\cdot\bar{z},\ H'(\rho\bar{z})\cdot\bar{z}\} \quad .$$

Note that F_n is continuous in its arguments. Define for $z = r\bar{z}(\theta) \in \mathbf{R}^{2N}$, $r = |z|$, $\bar{z}(\theta) = \frac{z}{|z|}$,

$$G_n(z) = \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma\in\Omega_n^{(1)}(\beta,\rho)} F_n(\rho,\bar{z}(\gamma))\,d\beta\,d\rho$$

and

$$\hat{H}_n(z) = \begin{cases} H(z), & \text{if } |z| \leq K_n, \\ G_n(z) + H(K_n\bar{z}(\theta)), & \text{if } K_n < |z| \ . \end{cases}$$

Note that when $r = |z| > K_{n+1}$, by (A.5) and (A.9),

$$G_n(z) = \int_{K_n}^{K_{n+1}} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma\in\Omega_n^{(1)}(\beta,\rho)} F_n(\rho,\bar{z}(\gamma))\,d\beta\,d\rho \tag{A.17}$$
$$+ \tau_n(r^{\mu\lambda} - K_{n+1}^{\mu\lambda}) \quad .$$

<u>Lemma A.18</u>. For $n \in \mathbf{N}$, $\hat{H} \in C^1(\mathbf{R}^{2N},\mathbf{R})$.

Proof. Since $H, M_n \in C^1(\mathbb{R}^{2N},\mathbb{R})$ and in the formula of G_n all the variables r, θ only appear linearly in the integration limits, $\hat{H}_n$ is C^1-continuous on $\{|z| < K_n\}$ and $\{|z| > K_n\}$. We only need to verify the C^1-continuity of $\hat{H}_n$ at every $\zeta \in \mathbb{R}^{2N}$ with $|\zeta| = K_n$, $\bar{\zeta} = \frac{\zeta}{|\zeta|}$.

For $z = r\bar{z}(\theta)$ with $r = |z| > K_n$, $\bar{z}(\theta) = \frac{z}{|z|}$,

$$\frac{\partial \hat{H}_n(z)}{\partial r} = \frac{\partial G_n(z)}{\partial r} = \frac{1}{V_n(r)} \int_{\Omega_n(\theta,r)} \min_{\gamma\in\Omega_n^{(1)}(\beta,r)} F_n(r,\bar{z}(\gamma))d\beta \tag{A.19}$$

$$= \min_{\gamma\in\Omega_n^{(1)}(\xi,r)} F_n(r,\bar{z}(\gamma)) ,$$

for some $\xi \in \Omega_n(\theta,r)$ by the mean value theorem of integration. Thus

$$\lim_{\substack{z\to\zeta \\ |z|>|\zeta|=K_n}} \frac{\partial \hat{H}_n(z)}{\partial r} = \lim_{\substack{z\to\zeta \\ |z|>|\zeta|=K_n}} \min_{\gamma\in\Omega_n^{(1)}(\xi,r)} F_n(r,\bar{z}(\gamma))$$

$$= F_n(K_n,\bar{\zeta}) = H'(K_n\bar{\zeta})\cdot\bar{\zeta}$$

$$= \frac{\partial H(\zeta)}{\partial r} .$$

From the definition of G_n we get that

$$\frac{\partial G_n(z)}{\partial \theta_1} = \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_{n,1}(\theta,\rho)} (1 - \frac{1}{2}\nu_n(\rho - K_n))\Big[\min_{\gamma\in\Delta_1} F_n(\rho,\bar{z}(\gamma)) \tag{A.20}$$

$$- \min_{\gamma\in\Delta_2} F_n(\rho,\bar{z})(\gamma))\Big]d\hat{\beta}_1 d\rho ,$$

where $\Delta_1 = \omega_{n,1}^{(1)}(\theta_1 + \frac{1}{2}(\pi - \theta_1)\nu_n(\rho - K_n),\rho) \times \Omega_{n,1}^{(1)}(\beta,\rho)$, $\Delta_2 = \omega_{n,1}^{(1)}(\theta_1 - \frac{1}{2}\theta_1\nu_n(\rho - K_n),\rho) \times \Omega_{n,1}^{(1)}(\beta,\rho)$. So for $K_n < r < K_n + \varepsilon_n$, we get that

$$\left|\frac{\partial G_n(z)}{\partial \theta_1}\right| < \frac{2}{\pi}\, 2\bar{M}_n \int_{K_n}^{r} \frac{1}{\sqrt{\rho - K_n}} d\rho \quad \text{(by (A.15))}$$

$$= \frac{8}{\pi}\bar{M}_n\sqrt{r - K_n} \to 0 \quad \text{as} \quad r \to K_n ,$$

where $\overline{M}_n = \max_{(\theta,\rho)\in\Omega\times[K_n,K_n+1]} F_n(\rho,\overline{z}(\theta))$. Thus

$$\lim_{\substack{z\to\zeta \\ |z|>|\zeta|=K_n}} \frac{\partial\hat{H}_n(z)}{\partial\theta_1} = \lim_{\substack{z\to\zeta \\ |z|>|\zeta|=K_n}} \left(\frac{\partial G_n(z)}{\partial\theta_1} + \frac{\partial H(K_n\overline{z}(\theta))}{\partial\theta_1}\right) = \frac{\partial H(\zeta)}{\partial\theta_1} .$$

Similarly we have that

$$\lim_{\substack{z\to\zeta \\ |z|>|\zeta|=K_n}} \frac{\partial\hat{H}_n(z)}{\partial\theta_i} = \frac{\partial H(\zeta)}{\partial\theta_i} \quad \text{for } i = 2,\ldots,2N-1 .$$

This completes the proof.

□

<u>Lemma A.21</u>. For $n \in \mathbf{N}$ and $z \in \mathbf{R}^{2N}$

$$\hat{H}_n(z) \le \hat{H}_{n+1}(z) \le H(z) . \tag{A.22}$$

<u>Proof</u>. 1°. We prove that $\hat{H}_n(z) \le H(z)$.

If $|z| \le K_n$, this is true since $\hat{H}_n(z) = H(z)$.

If $K_n < |z|$, write $z = r\overline{z}(\theta)$, then

$$\hat{H}_n(z) \le \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma\in\Omega_n^{(1)}(\beta,\rho)} H'(\rho\overline{z}(\gamma))\cdot\overline{z}(\gamma)d\beta d\rho$$

$$+ H(K_n\overline{z}(\theta)) .$$

By 1° of Lemma A.16, $\beta \in \Omega_n(\theta,\rho)$ implies that $\theta \in \Omega_n^{(1)}(\beta,\rho)$, so

$$\hat{H}_n(z) \le \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} H'(\rho\overline{z}(\theta))\cdot\overline{z}(\theta)d\beta d\rho$$

$$+ H(K_n\overline{z}(\theta))$$

$$= \int_{K_n}^{r} H'(\rho\overline{z}(\theta))\cdot\overline{z}(\theta)d\rho + H(K_n\overline{z}(\theta))$$

$$= H(z) .$$

2°. We prove that $\hat{H}_n(z) \le \hat{H}_{n+1}(z)$.

If $|z| < K_{n+1}$, this is a consequence of 1° since $\hat{H}_{n+1}(z) = H(z)$.

If $K_{n+1} < |z|$, write $z = r\bar{z}(\theta)$, then by the definition of $\hat{H}_n$,

$$\hat{H}_n(z) = \int_{K_{n+1}}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma\in\Omega_n^{(1)}(\beta,\rho)} F_n(\rho,\bar{z}(\gamma))d\beta d\rho$$

$$+ \hat{H}_n(K_{n+1}\bar{z}(\theta))$$

$$< \int_{K_{n+1}}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \max_{\gamma\in\Omega} M_n'(\rho\bar{z}(\gamma))\cdot\bar{z}(\gamma)d\beta d\rho$$

$$+ \hat{H}_n(K_{n+1}\bar{z}(\theta))$$

$$= \int_{K_{n+1}}^{r} \frac{1}{V_{n+1}(\rho)} \int_{\Omega_{n+1}(\theta,\rho)} \max_{\gamma\in\Omega} M_n'(\rho\bar{z}(\gamma))\cdot\bar{z}(\gamma)d\beta d\rho$$

$$+\hat{H}_n(K_{n+1}\bar{z}(\theta))$$

$$< \int_{K_{n+1}}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_{n+1}(\theta,\rho)} \min_{\gamma\in\Omega} F_{n+1}(\rho,\bar{z}(\gamma))d\beta d\rho$$

$$+ H(K_{n+1}\bar{z}(\theta)) \;,$$

here we used (A.9), (A.10), and 1° of this lemma. Thus

$$\hat{H}_n(z) < \hat{H}_{n+1}(z) \quad \text{if} \quad K_{n+1} < |z|$$

and this completes the proof.

□

Lemma A.23. For every $n \in \mathbb{N}$

$$\text{(A.24)} \qquad 0 < \mu\sigma\hat{H}_n(z) < \hat{H}_n'(z)\cdot z, \quad \forall |z| > r_0 \;.$$

Proof. Write $z = r\bar{z}(\theta)$, $r = |z|$, $\bar{z}(\theta) = \frac{z}{|z|}$.

If $r_0 < |z| < K_n$, (A.24) holds by (H2) and $\hat{H}_n(z) = H(z)$.

If $K_{n+1} < |z|$, then by the definition of $\hat{H}_n$ and (A.17) $\hat{H}_n'(z)\cdot z = \tau_n\mu\lambda r^{\mu\lambda}$. So

$$\hat{H}_n'(z)\cdot z = \mu\sigma\hat{H}_n(z) - \mu\sigma \int_{K_n}^{r} \frac{1}{V_n(\rho)} \int_{\Omega_n(\theta,\rho)} \min_{\gamma\in\Omega_n^{(1)}(\beta,\rho)} F_n(\rho,\overline{z}(\gamma))d\beta d\rho$$

$$- \mu\sigma H(K_n\overline{z}(\theta)) + \tau_n\mu\lambda r^{\mu\lambda}$$

$$> \mu\sigma\hat{H}_n(z) - \mu\sigma \int_{K_n}^{r} M_n'(\rho\overline{z}(\theta))\cdot\overline{z}(\theta)d\rho$$

$$- \mu\sigma H(K_n\overline{z}(\theta)) + \tau_n\mu\lambda r^{\mu\lambda}$$

(by 1° of Lemma A.16)

$$= \mu\sigma\hat{H}_n(z) + \tau_n\mu\lambda r^{\mu\lambda} - \mu\sigma\tau_n r^{\mu\lambda} + \mu\sigma M_n(K_n\overline{z}(\theta))$$

$$- \mu\sigma H(K_n\overline{z}(\theta))$$

$$> \mu\sigma\hat{H}_n(z) \quad ,$$

since $\lambda > \sigma$ and $M_n(K_n\overline{z}(\theta)) = H(K_n\overline{z}(\theta))$.

If $K_n < |z| < K_{n+1}$, by 2° of Lemma A.16,

$$\hat{H}_n'(z)\cdot z = \frac{r}{V_n(r)} \int_{\Omega_n(\theta,r)} \min_{\gamma\in\Omega_n^{(1)}(\beta,r)} F_n(r,\overline{z}(\gamma))d\beta$$

$$> \min_{\gamma\in\Omega_n^{(2)}(\theta,r)} rF_n(r,\overline{z}(\gamma))$$

$$= rF_n(r,\overline{z}(\xi)) \quad ,$$

for some $\xi \in \Omega_n^{(2)}(\theta,r)$, by the compactness of $\Omega_n^{(2)}(\theta,r)$. So we get that by 1° of Lemma A.16

$$\hat{H}_n'(z)\cdot z > \mu\sigma\hat{H}_n(z) + rF_n(r,\overline{z}(\xi)) - \mu\sigma \int_{K_n}^{r} F_n(\rho,\overline{z}(\theta))d\rho$$

(A.25)

$$- \mu\sigma H(K_n\overline{z}(\theta)) \quad .$$

We consider two cases:

<u>Case 1.</u> $F_n(r,\overline{z}(\xi)) = M_n'(r\overline{z}(\xi))\cdot\overline{z}(\xi)$.

Then from (A.25) and 1° of Lemma A.16

$$\hat{H}_n'(z)\cdot z > \mu\sigma\hat{H}_n(z) + M_n'(r\overline{z}(\xi))\cdot r\overline{z}(\xi)$$

$$- \mu\sigma \int_{K_n}^{r} M_n'(\rho\overline{z}(\theta))\cdot\overline{z}(\theta)d\rho - \mu\sigma H(K_n\overline{z}(\theta))$$

$$> \mu\sigma\hat{H}_n(z) + \mu\lambda M_n(r\overline{z}(\xi)) - \mu\sigma M_n(r\overline{z}(\theta))$$

(by (A.4) and (A.6))

$$> \mu\sigma\hat{H}_n(z) + \mu(\lambda-\sigma)M_n(r\overline{z}(\xi)) - \mu\sigma|M_n(r\overline{z}(\xi)) - M_n(r\overline{z}(\theta))|$$

$$> \mu\sigma\hat{H}_n(z) + \mu(\lambda-\sigma)\alpha_0 K_n^{\mu\lambda} - \mu\sigma|M_n(r\overline{z}(\xi)) - M_n(r\overline{z}(\theta))|$$

(by (A.7))

$$> \mu\sigma\hat{H}_n(z) \quad .$$

In the last inequality, we used (A.12), (A.13), (A.15), and that $\xi \in \Omega_n^{(2)}(\theta,r)$.

Case 2. $F_n(r,\overline{z}(\xi)) = H'(r\overline{z}(\xi))\cdot\overline{z}(\xi)$.

Then from (A.25) and 1° of Lemma A.16,

$$\hat{H}_n'(z)\cdot z > \mu\sigma\hat{H}_n(z) + H'(r\overline{z}(\xi))\cdot r\overline{z}(\xi) - \mu\sigma \int_{K_n}^{r} H'(\rho\overline{z}(\theta))\cdot\overline{z}(\theta)d\rho$$

$$- \mu\sigma H(K_n\overline{z}(\theta))$$

$$> \mu\sigma\hat{H}_n(z) + \mu H(r\overline{z}(\xi)) - \mu\sigma H(r\overline{z}(\theta)) \quad \text{(by (H2))}$$

$$> \mu\sigma\hat{H}_n(z) + \mu(1-\sigma)\alpha_0 K_n^{\mu} - \mu\sigma|H(r\overline{z}(\xi)) - H(r\overline{z}(\theta))|$$

(by (2.4))

$$> \mu\sigma\hat{H}_n(z) ,$$

here we used (A.12), (A.13), (A.15), and that $\xi \in \Omega_n^{(2)}(\theta,r)$.

Thus $\hat{H}_n'(z)\cdot z > \mu\sigma\hat{H}_n(z)$ if $K_n < |z| < K_{n+1}$.

Finally from (A.6) and (H2), $\hat{H}_n(z) > 0$ for $|z| > r_0$, and this completes the proof of (A.24).

□

To get 6° of Proposition 2.5, we modify $\hat{H}_n$'s again.

For $n \in \mathbb{N}$, $z = r\overline{z}(\theta)$, from (A.17) if $r > K_{n+1}$, we get that

$$\hat{H}_n(z) = G_n(K_{n+1}\bar{z}(\theta)) + \tau_n(r^{\mu\lambda} - K_{n+1}^{\mu\lambda}) + H(K_n\bar{z}(\theta)) \quad .$$

Set

$$C_n = \max_{z\in S^{2N-1}} |G_n(K_{n+1}z) + H(K_nz) - \tau_nK_{n+1}^{\mu\lambda}| + 1 \quad .$$

Then we have that

$$\text{(A.26)} \qquad \begin{aligned} \hat{H}_n(z) &\le \tau_n|z|^{\mu\lambda} + C_n \\ &\le (\tau_n + 1)|z|^{\mu\lambda} \quad \forall|z| \ge \max\{K_{n+1},C_n\} \end{aligned}$$

and

$$\text{(A.27)} \qquad \begin{aligned} \hat{H}_{n+1}(z) &\ge \tau_{n+1}|z|^{\mu\lambda} - C_{n+1} \\ &\ge (\tau_n + 1)|z|^{\mu\lambda} + |z|^{\mu\lambda} - C_{n+1} \quad \text{(by (A.1))} \\ &\ge (\tau_n + 1)|z|^{\mu\lambda} \quad \forall|z| \ge \max\{K_{n+2},C_{n+1}\} \quad . \end{aligned}$$

Let $\hat{K}_n = \max\{K_{n+2},C_n,C_{n+1}\}$ and define

$$H_n(z) = \chi(|z|, \hat{K}_n)\hat{H}_n(z) + (1 - \chi(|z|, \hat{K}_n))(\tau_n + 1)|z|^{\mu\lambda} \quad \forall z \in \mathbf{R}^{2N} \quad .$$

Then we have that $H_n \in C^1(\mathbf{R}^{2N},\mathbf{R})$ possesses the following properties,

$$\text{(A.28)} \qquad H_n(z) = \hat{H}_n(z) \quad \text{for} \quad |z| \le \hat{K}_n \ ,$$

$$\text{(A.29)} \qquad H_n(z) = (\tau_n + 1)|z|^{\mu\lambda} \quad \text{for} \quad |z| \ge \hat{K}_n + 1 \quad ,$$

$$\text{(A.30)} \qquad 0 < \mu\sigma H_n(z) \le H_n'(z)\cdot z \quad \forall|z| \ge r_0 \ .$$

From (A.26) - (A.29) and the definition of H_n we also have that

$$\text{(A.31)} \qquad \hat{H}_n(z) \le H_n(z) \le \hat{H}_{n+1}(z) \quad \forall z \in \mathbf{R}^{2N} \quad .$$

Now we can give the

<u>Proof of Proposition 2.5.</u> 1° - 3° are true from the definitions of K_n, $\hat{H}_n$, H_n and Lemma A.18. Lemma A.21 and (A.31) yield 4°. (A.30) gives 5°. 6° is a consequence of (A.29) by letting $\lambda_0 = \frac{\mu\lambda}{\mu\lambda-1}$.

The proof of Proposition 2.5 is complete.

□

REFERENCES

1. Bahri, A. and Berestycki, H. A perturbation method in critical point theory and applications. Trans. Amer. Math. Soc. 167 (1981), 1-32.

2. ________ and ____________. Forced vibrations of superquadratic Hamiltonian systems. Acta Math. 152 (1984), 143-197.

3. ________ and ____________. Existence of forced oscillations for some nonlinear differential equations. Comm. Pure and Appl. Math. 37 (1984), 403-442.

4. Benci, V. On critical points theory for indefinite functionals in the presence of symmetries. Trans. Amer. Math. Soc. 274 (1982), 533-572.

5. Benci, V. and Rabinowitz, P. H. Critical point theorems for indefinite functionals. Invent. Math. 52 (1979), 241-273.

6. Brezis, H. and Wainger, S. A note on limiting cases of Sobolev embeddings and convolution inequalities. Comm. P.D.E. 5(7), (1980), 773-789.

7. Dugundji, J. An extension of Tietze's theorem. Pacific J. Math. 1 (1951), 353-367.

8. Fadell, E. R., Husseini, S. and Rabinowitz, P. H. Borsuk-Ulam theorems for arbitrary S^1-actions and applications. Trans. Amer. Math. Soc. 174 (1982), 345-360.

9. Fadell, E. R. and Rabinowitz, P. H. Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. Invent. Math. 45 (1978), 139-174.

10. Krasnosel'skii, M. A. and Rutitsky, Y. B. Convex Functions and Orlicz Spaces. Hindustan Publ. Co. (India) Delhi (1962).

11. Long, Y. On the density of the range for some superquadratic operators. MRC Technical Summary Report #2859, University of Wisconsin-Madison (1985).

12. _______. Multiple solutions of perturbed superquadratic second order Hamiltonian systems. MRC Technical Summary Report #2963, University of Wisconsin-Madison (1987).

13. _______. Doctoral thesis, University of Wisconsin-Madison, 1987.

14. Natanson, I. P. Theory of Functions of a Real Variable, Vol. 1. Frederick Ungar Publ. Co. New York (1964). Revised Edition.

15. Pisani, R. and Tucci, M. Existence of infinitely many periodic solutions for a perturbed Hamiltonian system. Nonlinear Anal. T.M.&A. 8 (1984), 873-891.

16. Rabinowitz, P. H. Variational methods for nonlinear eigenvalue problems. In Eigenvalues of Nonlinear Problems. Roma (1974). Ediz. Cremonese.

17. ________________. Periodic solutions of Hamiltonian systems. Comm. Pure Appl. Math. 31 (1978), 157-184.

18. ________________. Multiple critical points of perturbed symmetric functionals. Trans. Amer. Math. Soc. 272 (1982), 753-769.

19. ________________. Periodic solutions of large norm of Hamiltonian systems. J. of Diff. Equa. 50 (1983), 33-48.

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